THE DIFFERENTIAL INVARIANTS OF GENERALIZED SPACES

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THE DIFFERENTIAL INVARIANTS OF GENERALIZED SPACES

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CONTENTS

CHAPTER I

N-DIMENSIONAL SPACES

CHAPTER II

AFFINE AND RELATED INVARIANTS

CHAPTER III

PROJECTIVE INVARIANTS

CHAPTER IV

CONFORMAL INVARIANTS

CHAPTER V

NORMAL COORDINATES

CHAPTER VI

SPATIAL IDENTITIES

CHAPTER VII

ABSOLUTE SCALAR DIFFERENTIAL INVARIANTS AND PARAMETERS

CHAPTER VIII

THE EQUIVALENCE PROBLEM

CHAPTER IX

REDUCIBILITY OF SPACES

CHAPTER X

FUNCTIONAL ARBITRARINESS OF SPATIAL INVARIANTS

PREFACE

The impetus given within the last decade and a half by the theory of relativity to the study of generalized spaces and their differential invariants has resulted in many interesting and important discoveries in this field. Most important among these and most fruitful for later developments was the concept of the infinitesimal parallel displacement formulated by Levi-Civita in 1917—a concept which was soon extended by Weyl in the theory of the general affinely connected space. The discovery of the existence of a projective theory of the affinely connected space was made in 1921 by Weyl. This was followed by the discovery of the affine representation of the projective theory as well as of the conformal theory of metric spaces by the author. More general geometrical theories of projective space, intimately related to this affine representation, were devised by Schouten, Veblen and others. Also L. P. Eisenhart arrived at the idea of the invariant theory of the group space of an r-parameter continuous group, which was more fully developed by Cartan and Schouten. These and other researches in the theory of the differential invariants of generalized spaces, or as it is sometimes called, the absolute differential calculus, mark the period through which we have just passed. Further interesting results in this field undoubtedly await the investigations of the future.

The following pages are intended to give the student a connected account, including the above recent developments, of the subject of the differential invariants of generalized spaces. It is my hope also that the book may be of some use to research workers. I have adopted the notation in most common use, which is undoubtedly the best. The book is exclusively analytical in character—above all I have striven for elegance of analytical procedure. Special geometrical points of view have been, as they should be, omitted, as such are primarily a matter of personal taste and should, in any case, be confined to books on geometry; I have thus excluded the viewpoint initiated by Cartan as well as the more recent geometrical formulations of Schouten and Veblen. While certain of the results given may possibly be included in the field of differential geometry, I have made no invasion of the strict province of this subject, i.e. the theory of curves and surfaces embedded in a space of higher dimensionality.

Among the features of the book the following may be mentioned. Chapter I contains a general discussion of the n-dimensional spaces which have won a recognized position in the literature. The immediately following chapters give the foundations of the invariant theories of these spaces, so arranged that their relationships and essential differences are exhibited. In Chapter V we have a discussion of normal coordinates in generalized spaces, which forms the basis of the general theory of extension. Chapter VI

contains a very complete exposition of spatial identities based on the fundamental concept of the complete set of identities of an invariant. The subject of absolute scalar differential invariants and parameters in generalized spaces, defined by means of complete systems of partial differential equations, is treated in Chapter VII. This requires some knowledge of the Lie theory of continuous groups and in this connection the underlying elements of the invariant theory of the group space are introduced. Then follow the important Chapters VIII and IX on the equivalence problem and the reducibility of spaces, and finally a chapter on the functional arbitrariness of spatial differential invariants.

It was decided after some reflection to make no attempt to present the theory of groups of motions in generalized spaces, since this leads to considerations of a topological nature entirely foreign to the local character of the subject of differential invariants.

Throughout the book the practice has been adopted of using *small print* for those passages which supplement the main ideas of the text. Such parts, however, are to be interpreted in no sense as being less important than those which are not so printed. At the end of each chapter there are to be found references to the original sources of the material contained in the chapter. This has required an occasional cross-reference, but on the whole has been deemed more satisfactory than placing all references at the end of the book; by bringing the references into closer proximity with the material of the text, the problem of making descriptive comments has been greatly facilitated.

During the preparation of the first four chapters of the manuscript, Dr E. W. Titt acted in the capacity of research assistant. After he left to take up his duties as a National Research Fellow, this work was undertaken by Dr N. H. Ball. Their tasks consisted in reading and checking the manuscript and in the preparation of the references; for this latter they are primarily responsible. To both these young men I extend my most hearty acknowledgments.

I wish finally to express my gratitude to the members of the Cambridge University Press for the many ways in which they have assisted during the printing of the book.

T. Y. THOMAS

March 1934

CHAPTER I

N-DIMENSIONAL SPACES

In this chapter we have presented the basic ideas of the important n-dimensional spaces which will be studied in detail in the subsequent chapters. An understanding of the summation convention and its uses as well as the idea of the ordinary vector* has been assumed, as such knowledge is usually in the possession of most readers approaching the more advanced treatment of the differential invariants of generalized spaces.

1. Space. Coordinates

By a space $\mathscr{V}$ of n dimensions we shall mean a set of elements or points for which one can define a system of sub-sets called neighbourhoods, satisfying the following conditions(1):

A. *The points of each neighbourhood V can be put into one to one reciprocal correspondence with the interior points of a hypersphere Σ of the Euclidean space of n dimensions.*

B. *Each point of the space belongs to at least one neighbourhood.*

C. *Let V be an arbitrary neighbourhood, Σ the hypersphere in correspondence with V, M a point of V, m the corresponding point of Σ and σ a hypersphere with centre m interior to Σ. There exists a neighbourhood V', of M, interior to V and such that the correspondents in Σ of all the points of V' belong to σ.*

D. *Let M be a point belonging to V, m its correspondent in Σ and V' a neighbourhood containing M. There exists a hypersphere σ of centre m such that the correspondents in V of all the points of Σ which belong to σ are in V'.*

E. *If M and N are two points, there exists two neighbourhoods V and V' containing M and N respectively and such that V and V' contain no point in common.*

A point A of the space $\mathscr{V}$ is said to be a *cluster point* for an infinite set of distinct points of this space if each neighbourhood containing A in its interior contains at least one point of the set distinct from A. We shall say that the space $\mathscr{V}$ is *open* or *closed* according as one can or cannot find an infinite set of points which admit no cluster point.

An infinite sequence of points $P_1, P_2, \ldots, P_n, \ldots$ is said to approach a limit point P if, given an arbitrary neighbourhood V containing P, a point P_n of the sequence can be found such that all following points are in V. It follows from Postulate E that an infinite sequence cannot admit two distinct limit points P and Q.

* This is, in fact, a special case of the affine tensor defined in § 10.

By a continuous curve is meant a set of points which can be put into one to one reciprocal correspondence with the numerical values of a real variable t satisfying $0 \leqq t \leqq 1$, such that if $t_n \to t_0$, the sequence of points corresponding to t_n tends toward the point corresponding to t_0. The space $\mathscr{T}$ is said to be connected if two arbitrary points of $\mathscr{T}$ can be joined by a continuous curve. In the study of differential invariants it is sufficient to limit our considerations to connected spaces $\mathscr{T}$.

By Postulates A, C, and D a one to one reciprocal continuous correspondence exists between the points of a neighbourhood V and the points interior to any hypersphere Σ. Hence the points of a neighbourhood V can be defined analytically by n real coordinates $x^1, \ldots, x^n$ such that if P is the limit point of a sequence P_n, the distance of P from P_n, as given by the Euclidean measure of distance, approaches zero as n approaches infinity. Any particular association of the coordinates $x^1, \ldots, x^n$ with the interior points of a neighbourhood V is called a *coordinate system*; more precisely this will be called the x coordinate system. As an abbreviation x^α will usually be written for the coordinates of a point P.

A coordinate system may exist which will cover the entire space $\mathscr{T}$; or it may be possible to find a coordinate system which will cover this space with the exception of one or more surfaces of lower dimensionality, the extent to which the space $\mathscr{T}$ can be covered by a single coordinate system depending obviously on its topological character. In the following we shall refer to the portion of the space $\mathscr{T}$ covered by the x coordinate system as the region $\mathscr{R}$. The coordinates of the region $\mathscr{R}$ thus constitute a class $\mathfrak{R}$ of sets of ordered real numbers $x^1, \ldots, x^n$ possessing the following properties: (1) if $a^1, \ldots, a^n$ belongs to the class $\mathfrak{R}$ so also does $x^1, \ldots, x^n$, where

$$|x^1 - a^1| < b^1, \quad \ldots, \quad |x^n - a^n| < b^n \tag{1.1}$$

for sufficiently small positive constants b^α, and (2) the points of $\mathscr{R}$ and the elements of $\mathfrak{R}$ are in one to one reciprocal correspondence. It is to be understood, however, that the extent to which the coordinate system covers the space $\mathscr{T}$ is immaterial. We shall, in fact, find it convenient from time to time to restrict the region $\mathscr{R}$ so that our results will be valid throughout the entire region $\mathscr{R}$; in conformity with the local nature of the subject of differential invariants this will constitute no essential loss of generality.

Rectangular Cartesian coordinates furnish a simple example of coordinates in the sense of the above definition. Another example is given by the polar coordinates of a Euclidean plane, provided that we exclude the polar axis including the origin, i.e. impose the conditions $r > 0$, $0 < \theta < 2\pi$. We might, of course, include the polar axis with the exception of the origin, by imposing the conditions $r > 0$, $0 \leqq \theta < 2\pi$; this would mean that we would sacrifice the existence of inequalities of the form (1.1) for all points of the space, although we would retain the property of a one to one reciprocal correspondence between points and coordinates. There is, however, an inherent simplicity in the coordinates as above defined which makes their use desirable in theoretical investigations.

The process by which we pass from one system of coordinates to another is called a *transformation of coordinates*. Such a transformation of coordinates is represented by a set of n equations

$$x^\alpha = f^\alpha(\bar{x}^1, \ldots, \bar{x}^n), \tag{1.2}$$

which expresses the fact that the coordinates x^α of a point P with respect to the x system of coordinates are determined when the coordinates $\bar{x}^\alpha$ of the same point P with respect to an $\bar{x}$ system of coordinates are given. Since a particular set of coordinates x^α determine a point P which in turn determines its set of coordinates $\bar{x}^\alpha$ with respect to the $\bar{x}$ system of coordinates, we have that the transformation (1.2) can be expressed in the inverse form

$$\bar{x}^\alpha = F^\alpha(x^1, \ldots, x^n). \tag{1.3}$$

If the values of the $\bar{x}^\alpha$ coordinates in terms of the x^α coordinates which are given by (1.3) are substituted into the right members of (1.2), these equations must be satisfied identically, i.e. they must reduce to the equations $x^\alpha = x^\alpha$, since otherwise there would be a relation between the independent quantities x^α. Similarly, the substitution of (1.2) into (1.3) must lead to equations which are satisfied identically. Hence (1.3) can be regarded as obtained from (1.2) by solving for the $\bar{x}^\alpha$ and (1.2) as obtained from (1.3) by solving for the x^α coordinates.

If the functions f^α and F^α are differentiable and possess finite derivatives at each point of $\mathscr{R}$, we shall have the set of identical relations

$$\frac{\partial x^\alpha}{\partial \bar{x}^\gamma} \frac{\partial \bar{x}^\gamma}{\partial x^\beta} = \delta^\alpha_\beta \tag{1.4}$$

holding throughout the region $\mathscr{R}$; here the Kronecker δ^α_β is defined as 1 or 0 according as $\alpha = \beta$, or $\alpha \neq \beta$, respectively. Taking the determinants of both members of (1.4), we obtain

$$\frac{\partial x^\alpha}{\partial \bar{x}^\gamma} \quad \frac{\partial \bar{x}^\gamma}{\partial x^\beta} = 1;$$

hence neither of these determinants can be zero at any point of the region $\mathscr{R}$.

The set of all transformations (1.2) of the coordinates of the region $\mathscr{R}$ obviously forms a *group* of which a *sub-group* is the set of all analytic transformations (1.2) of coordinates (see § 55). Much of what we shall have to say in the following work will require only the existence of a finite number of derivatives of orders one to p of the functions f^α; let us, nevertheless, for the sake of definiteness impose the condition of analyticity on these functions, i.e. limit ourselves to the group $\mathfrak{G}$ of analytic transformations (1.2) of the coordinates of $\mathscr{R}$ with analytic inverses (1.3). Cases for which this requirement is unnecessarily drastic will be found to be discerned easily and need be the occasion of no especial comment.*

* The statement that the functions f^α are analytic in the region $\mathscr{R}$ means that each of these functions can be expanded in a convergent power series about an arbitrary point P of $\mathscr{R}$.

2. Affine connection

A *structure* can be imposed on the point space of § 1 by establishing a correspondence between the totality of vectors at a point P and the totality of vectors at an infinitely nearby point Q: *corresponding vectors at P and Q will be said to be parallel*. A vector at Q which corresponds to a vector at P will be said to result from the vector at P by *infinitesimal parallel displacement*(2). When this law of correspondence is such that the change in the components of a vector by infinitesimal parallel displacement is (1) linear in the components of the vector and (2) linear in the components dx^{β} of the displacement $P \rightarrow Q$, we say that the point P is *affinely connected* with the point Q or simply that the space bears an *affine connection* (cf. § 10). The change $d\xi^{\alpha}$ in the components of a contravariant vector due to an infinitesimal parallel displacement from P to Q in a space with affine connection is therefore given by an expression of the form

$$d\xi^{\alpha} = -L^{\alpha}_{\beta\gamma}\xi^{\beta}dx^{\gamma}. \tag{2.1}$$

By imposing the condition that the scalar product $\xi^{\alpha}\mu_{\alpha}$ is invariant under an infinitesimal parallel displacement, the above quantities $L^{\alpha}_{\beta\gamma}$ are likewise introduced as the coefficients of the set of bilinear forms which define the change in the components of a covariant vector as the result of an infinitesimal parallel displacement. Thus

$$d(\xi^{\alpha}\mu_{\alpha}) = \xi^{\alpha}d\mu_{\alpha} + \mu_{\alpha}d\xi^{\alpha} = \xi^{\alpha}(d\mu_{\alpha} - L^{\beta}_{\alpha\gamma}\mu_{\beta}dx^{\gamma}) = 0,$$

and since this must hold for arbitrary components ξ^{α}, we have

$$d\mu_{\alpha} = L^{\beta}_{\alpha\gamma}\mu_{\beta}dx^{\gamma}. \tag{2.2}$$

The quantities $L^{\alpha}_{\beta\gamma}$ are called the *components of affine connection*; they will be assumed to be arbitrary analytic functions in the region $\mathscr{R}$ covered by our coordinate system.

The above concept of infinitesimal parallel displacements can be extended to permit the comparison as to parallelism of vectors at distant points P and Q of the region $\mathscr{R}$ provided that a route of displacement from P to Q is specified. Let the curve C be defined by the parametric equations $x^{\alpha} = \phi^{\alpha}(s)$ and consider the equations

$$d\xi^{\alpha} = -L^{\alpha}_{\beta\gamma}\xi^{\beta}\frac{dx^{\gamma}}{ds}, \tag{2.3}$$

valid along C. A solution $\xi^{\alpha}(s)$ of these equations, determined by the arbitrary initial conditions $\xi^{\alpha}(0)$ in accordance with the general existence theorem for systems of linear differential equations, defines a vector at each point of C: we say of these vectors that they are *parallel with respect to the curve C* and that they are generated by parallel displacement along C of the vector with components $\xi^{\alpha}(0)$ at the point P. Parallel displacement of this latter vector along different routes C and C' joining points P and Q will in

general result in different vectors at Q (see Fig. 1). Similar remarks apply to the parallel displacement of covariant vectors.*

The components of affine connection $L^{\alpha}_{\beta\gamma}$ can be broken up into two parts by putting

$$L^{\alpha}_{\beta\gamma}=\Gamma^{\alpha}_{\beta\gamma}+\Omega^{\alpha}_{\beta\gamma},$$

such that

$$\text{(2.4)} \qquad (a)\ \Gamma^{\alpha}_{\beta\gamma}=\tfrac{1}{2}(L^{\alpha}_{\beta\gamma}+L^{\alpha}_{\gamma\beta}), \quad (b)\ \Omega^{\alpha}_{\beta\gamma}=\tfrac{1}{2}(L^{\alpha}_{\beta\gamma}-L^{\alpha}_{\gamma\beta}).$$

If the quantities Ω vanish identically in the region $\mathscr{R}$, the affine connection is said to be *symmetric*; otherwise it is said to be *asymmetric*. In the following we shall refer to the $\Gamma^{\alpha}_{\beta\gamma}$ as the components of the symmetric affine connection or in fact merely as the components of affine connection when there is no danger of ambiguity.

The following gives a geometrical property connected with an affine connection. Let the point P have coordinates x^{α} and consider two infinitesimal vectors d and δ at P with components dx^{α} and δx^{α} respectively. Denote the points with components

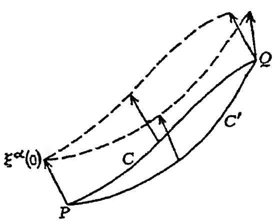

Fig. 1.

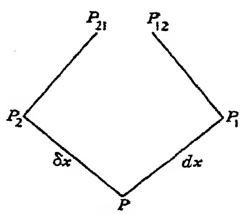

Fig. 2.

$x^{\alpha}+dx^{\alpha}$ and $x^{\alpha}+\delta x^{\alpha}$ by P_1 and P_2 respectively. When the vector d is subjected to a parallel displacement to the point P_2, the coordinates of the end point of the vector at P_2, namely P_{21}, are given by

$$x^{\alpha}+\delta x^{\alpha}+dx^{\alpha}-L^{\alpha}_{\beta\gamma}\,dx^{\beta}\,\delta x^{\gamma}$$

when terms of higher order are neglected. Similarly, when the vector δ is transported

* An extension of the definition of parallelism with respect to a curve C can be made, which will permit us to say that $\eta^{\alpha}(s)=a(s)\,\xi^{\alpha}(s)$, where $a(s)$ is an arbitrary analytic function of s, is parallel with respect to C if $\xi^{\alpha}(s)$ is parallel with respect to the curve C. We have

$$(a) \qquad \frac{d\eta^{\alpha}}{ds}+L^{\alpha}_{\beta\gamma}\eta^{\beta}\frac{dx^{\gamma}}{ds}=\eta^{\alpha}b(s),$$

$$(b) \qquad b(s)=\frac{d\log a}{ds},$$

and hence

$$(c) \qquad \frac{\dfrac{d\eta^{\alpha}}{ds}+L^{\alpha}_{\beta\gamma}\eta^{\beta}\dfrac{dx^{\gamma}}{ds}}{\eta^{\alpha}}=\frac{\dfrac{d\eta^{\rho}}{ds}+L^{\rho}_{\mu\nu}\eta^{\mu}\dfrac{dx^{\nu}}{ds}}{\eta^{\rho}}.$$

Let us say that the vectors with components $\eta^{\alpha}(s)$ are parallel with respect to the curve C if equations (c) are satisfied. Conversely, if equations (c) are satisfied along C we can pass to equations (a) and hence to the determination of a function $a(s)$ such that the components $\xi^{\alpha}(s)$ defined by $\eta^{\alpha}=a\xi^{\alpha}$ satisfy (2.3) along C. See L. P. Eisenhart, "Fields of parallel vectors in the geometry of paths", *Proc. N.A.S.* **8** (1922), pp. 207–12.

by parallel displacement to the point P_1 the coordinates of its end point P_{12} are given by

$$x^\alpha + dx^\alpha + \delta x^\alpha - L^\alpha_{\beta\gamma}\,\delta x^\beta\, dx^\gamma.$$

Hence a necessary and sufficient condition that the points P_{21} and P_{12} coincide for arbitrary vectors d and δ is that the affine connection $L^\alpha_{\beta\gamma}$ be symmetric.

3. Affine geometry of paths

A curve C defined by $x^\alpha = \phi^\alpha(s)$ which possesses the property that its tangents are parallel with respect to the curve C itself is given as a solution of the equations

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0; \tag{3.1}$$

such curves are a generalization of the straight lines of Euclidean space. They will be referred to as *paths* and may be thought of as affording a means by which one can find his way about in a space of affine connection. The body of theorems which state properties of paths as defined by a particular set of equations (3.1) will be called *an affine geometry of paths* (3). As an example of an affine property we may mention the property of a set of vectors in virtue of which they are mutually parallel with respect to an arbitrary curve C.

An important property of the paths is the following: For any point P of the region $\mathscr{R}$ there exists a domain $\mathscr{D}$ containing P such that any point Q of $\mathscr{D}$ is joined to P by one and only one path C lying in the domain $\mathscr{D}$.* If we differentiate the equations (3.1) successively, we obtain the following sequence

$$\frac{d^3x^\alpha}{ds^3} + \Gamma^\alpha_{\beta\gamma\delta}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}\frac{dx^\delta}{ds} = 0,$$

$$\frac{d^4x^\alpha}{ds^4} + \Gamma^\alpha_{\beta\gamma\delta\epsilon}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}\frac{dx^\delta}{ds}\frac{dx^\epsilon}{ds} = 0,$$

....

The coefficients Γ in these equations are given by the recurrence formula

$$\Gamma^\alpha_{\beta\gamma\ldots\mu\nu} = \frac{1}{M}P\left[\frac{\partial\Gamma^\alpha_{\beta\gamma\ldots\mu}}{\partial x^\nu} - \Gamma^\alpha_{\sigma\gamma\ldots\mu}\Gamma^\sigma_{\beta\nu} - \ldots - \Gamma^\alpha_{\beta\gamma\ldots\sigma}\Gamma^\sigma_{\mu\nu}\right],$$

where M denotes the number of subscripts $\beta, \gamma, \ldots, \mu, \nu$ and the symbol P denotes the sum of the terms obtainable from the ones inside the brackets by permuting the set of subscripts cyclically. As so defined the above functions Γ have the property of being unchanged by any permutation of

* More generally it can be shown that for any point P of the region $\mathscr{R}$ there exists a domain $\mathscr{D}$ containing P such that any two points U and V of $\mathscr{D}$ can be joined by one and only one path C lying in the domain $\mathscr{D}$. See J. H. C. Whitehead, "Convex regions in the geometry of paths" *Quart. Journ. of Math.* 3 (1932), pp. 33–42.

the subscripts. The parametric equations $x^\alpha = \phi^\alpha(s)$ of a path C are determined by (3.1) and the above sequence of equations in conjunction with the initial values $x^\alpha = p^\alpha$ and $dx^\alpha/ds = \xi^\alpha$ corresponding to the value $s=0$ of the parameter. We have in fact

$$\phi^\alpha(s) \equiv p^\alpha + \xi^\alpha s - \frac{1}{2!}\Gamma^\alpha_{\beta\gamma}(p)\,\xi^\beta\xi^\gamma s^2 - \frac{1}{3!}\Gamma^\alpha_{\beta\gamma\delta}(p)\,\xi^\beta\xi^\gamma\xi^\delta s^3 - \dots,$$

where the right members are convergent power series for sufficiently small values of s. A path C is therefore uniquely determined by the specification of a point P and a "direction" with components ξ^α through P. Putting $y^\alpha = \xi^\alpha s$, the above series give

$$x^\alpha = p^\alpha + y^\alpha - \frac{1}{2!}\Gamma^\alpha_{\beta\gamma}(p)\,y^\beta y^\gamma - \frac{1}{3!}\Gamma^\alpha_{\beta\gamma\delta}(p)\,y^\beta y^\gamma y^\delta - \dots, \tag{3.2}$$

and these series will converge in a domain $\mathscr{N}$ defined by $|y^\alpha| < a^\alpha$, where the a's are sufficiently small positive constants. Since the Jacobian determinant of the right members of (3.2) with respect to the variables y^α is equal to unity at the point P, these equations can be solved so as to obtain

$$y^\alpha = x^\alpha - p^\alpha + \Lambda^\alpha(p, x-p),$$

where Λ^α is a multiple power series in $x^\alpha - p^\alpha$ beginning with second order terms. Hence (3.2) defines a coordinate transformation to a system of coordinates y^α such that the equations $y^\alpha = \xi^\alpha s$, where the ξ's are arbitrary constants, represent a path through the origin of this system, i.e. the point P; conversely any path through the point P can be so represented. In consequence of this fact the coordinates y^α are called *normal coordinates* (4). Through any point Q with coordinates q^α in the domain $\mathscr{N}$ defined by $|y^\alpha| < a^\alpha$, a path can be drawn to the origin, for example $y^\alpha = q^\alpha s$ are the equations of such a path and this path will lie entirely in $\mathscr{N}$; as there is evidently one and only one such path through the origin and the point Q, the above statement is proved.*

* It has been shown by Douglas that a general system of curves for which the above local property is satisfied, and also such that one and only one curve passes through any point in a given direction, is defined as solutions of the differential equations

$$\frac{d^2x^\alpha}{ds^2} = H^\alpha(x, p), \quad p^\alpha = \frac{dx^\alpha}{ds},$$

where the functions H^α are homogeneous of the second degree in p. See J. Douglas, "The general geometry of paths", *Ann. of Math.* (2), **29** (1927), pp. 143–68. This extension of the geometry of paths, as above defined, brings this theory into relation with the work on general metric or Finsler spaces, in which connection the following authors may be consulted:

P. Finsler, *Ueber Kurven und Flächen in allgemeinen Räumen*, Dissertation, Göttingen, 1918.

J. L. Synge, "A generalization of the Riemannian line element", *Trans. Amer. Math. Soc.* **27** (1925), pp. 61–7.

J. H. Taylor, "A generalization of Levi-Civita's parallelism and the Frenet formulas", *Trans. Amer. Math. Soc.* **27** (1925), pp. 246–64.

L. Berwald, "Ueber Parallelübertragung in Räumen mit allgemeiner Massbestimmung", *Jahresb. Deutsch. Math. Ver.* **34** (1925–6), pp. 213–20; "Untersuchung über die Krümmung allgemeiner metrischer Räume...", *Math. Zeit.* **25** (1926), pp. 40–73; "Ueber zweidimensionale allgemeine metrische Räume", *Crelle*, **156** (1927), pp. 191–210.

A further extension in this direction has also been given by J. Douglas, "Systems of K-dimensional manifolds in an N-dimensional space", *Math. Ann.* **105** (1931), pp. 707–33.

If, under the transformation (3.2) the components of affine connection

$$\Gamma^{\alpha}_{\beta\gamma}(x) \to C^{\alpha}_{\beta\gamma}(y),$$

then, at $y^\alpha = 0$, the components $C^{\alpha}_{\beta\gamma}$ are equal to zero (see § 35 in Chapter V). Hence the change $d\xi^\alpha$ or $d\mu_\alpha$ in the components of a vector defined by the above symmetric affine connection vanishes for infinitesimal parallel displacements from the origin of a normal coordinate system to any infinitely nearby point Q.

The differential equations of a path C can be written in a form which is independent of the particular parameter s occurring in (3.1). To deduce such a form of the equation of the paths let us make an arbitrary analytic substitution of parameter $s \to t$, as a result of which the parametric equations of a path C become $x^\alpha = \psi^\alpha(t)$. These latter functions satisfy the differential equations

$$\frac{\dfrac{d^2x^\alpha}{dt^2} + \Gamma^{\alpha}_{\beta\gamma}\dfrac{dx^\beta}{dt}\dfrac{dx^\gamma}{dt}}{\dfrac{dx^\alpha}{dt}} = \frac{d^2t}{\left(\dfrac{dt}{ds}\right)^2} \tag{3.3}$$

Hence the equations(5)

$$\frac{\dfrac{d^2x^\alpha}{dt^2} + \Gamma^{\alpha}_{\beta\gamma}\dfrac{dx^\beta}{dt}\dfrac{dx^\gamma}{dt}}{\dfrac{dx^\alpha}{dt}} = \frac{\dfrac{d^2x^\rho}{dt^2} + \Gamma^{\rho}_{\mu\nu}\dfrac{dx^\mu}{dt}\dfrac{dx^\nu}{dt}}{\dfrac{dx^\rho}{dt}} \tag{3.4}$$

are satisfied by the equations of the paths and are such that they continue to be satisfied if the independent variable in the parametric representation of any path is subjected to an arbitrary transformation.

It is evident from (3.3) that the differential equations (3.1) will continue to be satisfied if the independent variable s in the equations $x^\alpha = \phi^\alpha(s)$ of a path is replaced by $as + b$, where a and b are constants; also it is evident that $t = as + b$ is the most general substitution which will leave the form of the equations (3.1) invariant.

The system of curves defined by (3.4) is, moreover, no more general than that defined by (3.1). For, suppose that $x^\alpha = \psi^\alpha(t)$ satisfies (3.4) and let any of the expressions whose equality is asserted by (3.4) be represented by $\Phi(t)$. Then

$$\frac{\dfrac{d^2x^\alpha}{dt^2} + \Gamma^{\alpha}_{\beta\gamma}\dfrac{dx^\beta}{dt}\dfrac{dx^\gamma}{dt}}{\dfrac{dx^\alpha}{dt}} = \Phi(t).$$

Making a transformation of parameter $t \to s$, these latter equations become

$$\frac{\dfrac{d^2x^\alpha}{ds^2} + \Gamma^{\alpha}_{\beta\gamma}\dfrac{dx^\beta}{ds}\dfrac{dx^\gamma}{ds}}{dx^\alpha} = \frac{\Phi(t)}{\left(\dfrac{ds}{dt}\right)} - \frac{\dfrac{d^2s}{dt^2}}{\left(\dfrac{ds}{dt}\right)^2}. \tag{3.5}$$

Now if
$$s=f(t)=A+B\int e^{\int\Phi(t)\,dt}\,dt,$$
where A and B are constants, equations (3.5) reduce to (3.1). Hence the equations $x^\alpha=\psi^\alpha(t)$ of any curve defined by (3.4) can be written as solutions of (3.1). In other words, *equations* (3.1) *can be replaced by* (3.4) *as the equations which define the system of paths C.*

4. Projective geometry of paths

Let us inquire under what circumstances a set of differential equations

$$\frac{d^2x^\alpha}{ds^2}+\Lambda^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}=0 \tag{4.1}$$

can represent the same paths as (3.1). Suppose that a curve $x^\alpha=\psi^\alpha(t)$ is a path both for (3.1) and (4.1). The functions $\psi^\alpha(t)$ are not necessarily solutions of (3.1) or (4.1), but they are solutions of (3.4) and also of the corresponding equations determined by (4.1), i.e.

$$\frac{dx^\rho}{dt}\left(\frac{d^2x^\alpha}{dt^2}+\Lambda^\alpha_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt}\right)=\frac{dx^\alpha}{dt}\left(\frac{d^2x^\rho}{dt^2}+\Lambda^\rho_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right). \tag{4.2}$$

Between (3.4) and (4.2) we can eliminate the second derivatives, obtaining

$$\frac{(\Gamma^\alpha_{\beta\gamma}-\Lambda^\alpha_{\beta\gamma})}{\dfrac{dx^\alpha}{dt}}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt}=\frac{(\Gamma^\rho_{\beta\gamma}-\Lambda^\rho_{\beta\gamma})}{\dfrac{dx^\rho}{dt}}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt}. \tag{4.3}$$

Let
$$\Gamma^\alpha_{\beta\gamma}-\Lambda^\alpha_{\beta\gamma}=\phi^\alpha_{\beta\gamma},\quad \phi^\alpha_{\alpha\beta}=(n+1)\,\phi_\beta.$$
Then equations (4.3) can be written

$$(\phi^\alpha_{\beta\gamma}\delta^\rho_\epsilon-\phi^\rho_{\beta\gamma}\delta^\alpha_\epsilon)\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt}\frac{dx^\epsilon}{dt}=0,$$

and, since dx^α/dt may be chosen arbitrarily, this gives

$$\phi^\alpha_{\beta\gamma}\delta^\rho_\epsilon+\phi^\alpha_{\gamma\epsilon}\delta^\rho_\beta+\phi^\alpha_{\epsilon\beta}\delta^\rho_\gamma=\phi^\rho_{\beta\gamma}\delta^\alpha_\epsilon+\phi^\rho_{\gamma\epsilon}\delta^\alpha_\beta+\phi^\rho_{\epsilon\beta}\delta^\alpha_\gamma.$$

Setting $\rho=\epsilon$ in these latter equations and summing with respect to ϵ, we obtain

$$\Gamma^\alpha_{\beta\gamma}-\Lambda^\alpha_{\beta\gamma}=\phi_\beta\delta^\alpha_\gamma+\phi_\gamma\delta^\alpha_\beta. \tag{4.4}$$

If equations (3.1) and (4.1) are to represent the same system of paths, the above relations (4.4) must therefore be satisfied.

Conversely let ϕ_α represent a set of n functions of the coordinates x^α, analytic in the region $\mathscr{R}$, and consider the differential equations (3.1) and (4.1) subject to the relations (4.4). From these latter relations we see that along any path C determined by (3.1) we have

$$2\phi_\rho\frac{dx^\rho}{dt}=\frac{(\Gamma^\alpha_{\beta\gamma}-\Lambda^\alpha_{\beta\gamma})}{\dfrac{dx^\alpha}{dt}}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt}$$

Hence (4.3) is satisfied. But if (4.3) is subtracted from (3.4), the corresponding equations in Λ are obtained. Every path with respect to the Γ's is therefore a path with respect to the Λ's. Hence *a necessary and sufficient condition that* (3.1) *and* (4.1) *shall represent the same system of paths is that a set of functions* ϕ_α *exist such that the relations* (4.4) *are satisfied.*

The body of theorems which states properties of the paths independently of any particular definition of affine connection Γ is called the *projective geometry of paths* (6). Two symmetric affine connections whose components $\Gamma^\alpha_{\beta\gamma}$ and $\Lambda^\alpha_{\beta\gamma}$ are related by (4.4) will be said to be obtainable from one another by a *projective change* of affine connection; the projective geometry of paths is therefore concerned with properties of the paths which remain unaltered under projective changes of the affine connection Γ. Evidently such a property of the paths is furnished by the result obtained in § 3, namely that a point P of the region $\mathcal{R}$ is contained in a domain $\mathcal{N}$ such that any point Q of $\mathcal{N}$ is joined to P by one and only one path C lying in the domain $\mathcal{N}$. This result therefore constitutes a theorem in the projective geometry of paths.

If we put $\alpha=\beta$ in (4.4) and then sum on the index β, we obtain a set of equations which can be solved for the quantities ϕ_γ; elimination of the ϕ_γ from (4.4) by means of these latter equations gives

$$\Lambda^\alpha_{\beta\gamma}-\frac{\delta^\alpha_\beta}{n+1}\Lambda^\sigma_{\sigma\gamma}-\frac{\delta^\alpha_\gamma}{n+1}\Lambda^\sigma_{\sigma\beta}=\Gamma^\alpha_{\beta\gamma}-\frac{\delta^\alpha_\beta}{n+1}\Gamma^\sigma_{\sigma\gamma}-\frac{\delta^\alpha_\gamma}{n+1}\Gamma^\sigma_{\sigma\beta}.$$

Hence the set of quantities $\Pi^\alpha_{\beta\gamma}$ defined by

$$\Pi^\alpha_{\beta\gamma}=\Gamma^\alpha_{\beta\gamma}-\frac{\delta^\alpha_\beta}{n+1}\Gamma^\sigma_{\sigma\gamma}-\frac{\delta^\alpha_\gamma}{n+1}\Gamma^\sigma_{\sigma\beta}$$

constitute the components of a connection which remains unaltered by all projective changes of the affine connection Γ. The connection with components $\Pi^\alpha_{\beta\gamma}$ is called the *projective connection*. If we write the equations of the paths in the form

$$\frac{d^2x^\alpha}{dp^2}+\Pi^\alpha_{\beta\gamma}\frac{dx^\beta}{dp}\frac{dx^\gamma}{dp}=0, \tag{4.5}$$

we observe that the parameter p is normalized in the sense that it is independent of projective changes of the affine connection Γ; this parameter p is called the *projective parameter* (7). The theory of the projective connection and projective parameter evidently belongs to the projective geometry of paths.

5. Riemann or metric space

The concept of distance does not enter into the preceding considerations. We now introduce this concept by the assumption that the distance ds between two infinitely nearby points P and Q is given by

$$ds^2=g_{\alpha\beta}dx^\alpha dx^\beta, \tag{5.1}$$

where the right member of this equation is a positive definite quadratic form in the coordinate differences dx^α of the points P, Q; the coefficients $g_{\alpha\beta}$ are analytic functions of the coordinates x^α of the region $\mathscr{R}$. The above form will be called the *fundamental quadratic form*. Since the form (5.1) is positive definite, it follows that the determinant g of its coefficients is positive, i.e. explicitly

$$(5.2) \qquad \begin{vmatrix} g_{11} & \cdots & g_{1n} \\ & & \\ g_{n1} & \cdots & g_n \end{vmatrix} > 0$$

throughout $\mathscr{R}$.* A space bearing the above structure is known as a *Riemann space*(8). The structure of a Riemann space not only permits the determination of lengths of curves, angles, areas, volumes, etc., but likewise leads to a definition of infinitesimal parallel displacement as described in § 2.

(*a*) *Length of a curve*. Consider the curve C: $x^\alpha = \phi^\alpha(t)$ in the region $\mathscr{R}$. The length of C from the point A, with coordinates x^α corresponding to $t = t_0$, to the variable point P, of coordinates x^α determined by an arbitrary value of the parameter t, is by definition

$$(5.3) \qquad s = \int_{t_0}^{t} \sqrt{g_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}}\,dt.$$

This latter equation can, moreover, be regarded as a parameter transformation $t \to s$ leading to the parametric representation of the curve C with parameter s denoting the distance of the variable point P of C from the above fixed point A. As defined by (5.3), the length of the curve C is easily seen to be independent of the parameterization.

(*b*) *Length of a vector*. By definition the square of the length of a contravariant vector with components ξ^α is $g_{\alpha\beta}\xi^\alpha\xi^\beta$. Similarly the square of the length of a covariant vector having components μ_α is $g^{\alpha\beta}\mu_\alpha\mu_\beta$, where

$$3 = \frac{\text{cofactor of } g_{\alpha\beta} \text{ in } g}{}$$

* By definition a positive definite quadratic form $\phi = g_{\alpha\beta}x^\alpha x^\beta$ in n variables $x^1, \ldots, x^n$ is positive for real values of the variables $x^1, \ldots, x^n$ not all of which are zero, and vanishes only for

$$x^1 = \ldots = x^n = 0;$$

a negative definite quadratic form is defined in an analogous manner.

If the above form ϕ is definite it follows that the determinant g is different from zero; in particular if the form ϕ is positive definite, we have $g > 0$. To prove this let us suppose that ϕ is definite and let us consider the equations

$$g_{\alpha\beta}x^\beta = 0 \quad (\alpha = 1, \ldots, n).$$

Now if $g = 0$, these equations can be solved so as to obtain a real set of values $x^1, \ldots, x^n$ not all of which are zero; for these values the form ϕ will vanish contrary to hypothesis. If ϕ is positive definite, it can be reduced to the sum of squares

$$(y^1)^2 + \ldots + (y^n)^2$$

by a real linear substitution of the form $x^\alpha = a^\alpha_\sigma y^\sigma$, the determinant $|a^\alpha_\beta|$ of which does not vanish. Hence $g_{\alpha\beta}a^\alpha_\sigma a^\beta_\tau = \delta^\sigma_\tau$ and from this it follows that $g\,|a^\alpha_\sigma|^2 = 1$. In other words $g > 0$.

or

$$g_{\alpha\beta}g^{\alpha\gamma}=\delta^{\gamma}_{\beta}, \tag{5.4}$$

these latter equations being equivalent to the former. A vector of unit length is said to be a *unit vector*.

(c) *Angle between two directions. Orthogonality.* The angle θ between the two directions d and δ, defined by the infinitesimal displacements dx^α and δx^α respectively, is given by the equation

$$\cos\theta=\frac{g_{\alpha\beta}dx^\alpha\delta x^\beta}{\sqrt{g_{\mu\nu}dx^\mu dx^\nu}\sqrt{g_{\sigma\tau}\delta x^\sigma\delta x^\tau}}. \tag{5.5}$$

If $\cos\theta=0$, the two displacements d and δ are said to be *orthogonal*.

To show that the above formula for $\cos\theta$ is legitimate, we must show that $|\cos\theta|\leqq 1$. Let us suppose first that the two displacements d and δ are proportional so that we can put $\delta x^\alpha=b\,dx^\alpha$, where b is a positive or negative constant. Then (5.5) gives $\cos\theta=1$ for b positive and $\cos\theta=-1$ for b negative; hence in the first case the angle between the displacements is zero, and in the second case it is equal to two right angles. Now assume that the displacements d and δ are not proportional, i.e. that we cannot find constants μ and ν, not both zero, such that $\mu dx^\alpha+\nu\delta x^\alpha$ vanishes. Then make the substitution $z^\alpha=\mu dx^\alpha+\nu\delta x^\alpha$ in the quadratic form $g_{\alpha\beta}z^\alpha z^\beta$ which therefore becomes

$$g_{\alpha\beta}z^\alpha z^\beta=\mu^2\phi(d,d)+2\mu\nu\phi(d,\delta)+\nu^2\phi(\delta,\delta),$$

where

$$\phi(d,d)=g_{\alpha\beta}dx^\alpha dx^\beta,$$
$$\phi(d,\delta)=g_{\alpha\beta}dx^\alpha\delta x^\beta,$$
$$\phi(\delta,\delta)=g_{\alpha\beta}\delta x^\alpha\delta x^\beta.$$

Since the z^α cannot all vanish for μ, ν, not both equal to zero, it follows that

$$\mu^2\phi(d,d)+2\mu\nu\phi(d,\delta)+\nu^2\phi(\delta,\delta)>0, \tag{5.6}$$

provided that the real constants μ, ν do not both vanish; hence

$$\phi(d,d)\,\phi(\delta,\delta)-\phi^2(d,\delta)>0,$$

since otherwise it would be possible to find real constants μ, ν not both zero such that the left member of (5.6) would be equal to zero. Substitution of the above quadratic forms for the ϕ in this last inequality shows immediately that the absolute value of the right member of (5.5) is less than unity.

Since the dx^α and δx^α are the components of contravariant vectors, the above formula (5.5) can be considered to define the angle between two contravariant vectors provided that the components of neither of these vectors all vanish. In particular the angle θ between two unit vectors having components ξ^α and ζ^α is given by

$$\cos\theta=g_{\alpha\beta}\xi^\alpha\zeta^\beta.$$

(d) *Volume of space.* The volume of an n-dimensional portion of the region $\mathscr{R}$ is defined by the integral

$$\int\int\cdots\int\sqrt{g}\,dx^1dx^2\ldots dx^n, \tag{5.7}$$

where the integration is extended over the portion of the space in question.

(*e*) *Geodesic curves*. Consider a curve C joining any two points A and B of the region $\mathscr{R}$ (see Fig. 3); let the curve C have as its equations

$$x^\alpha = \phi^\alpha(s), \quad a \leqq s \leqq b,$$

where s denotes the arc length measured from the fixed point A. Also consider the one parameter family of infinitely nearby curves which likewise join the points A and B and which have equations

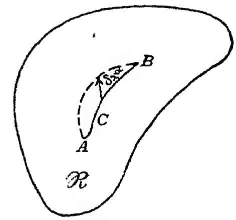

Fig. 3.

$$x^\alpha = \psi^\alpha(s, \epsilon), \quad a \leqq s \leqq b,$$

where the parameter ϵ takes on values in the neighbourhood of $\epsilon = 0$ and the function ψ^α is such that $\psi^\alpha(s, 0) = \phi^\alpha(s)$. Since all curves of this family pass through the points A and B, it follows that $\psi^\alpha(a, \epsilon)$ and $\psi^\alpha(b, \epsilon)$ are independent of the parameter ϵ. The length of any curve of the family is given by

$$l(\epsilon) = \int_a^b \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}}\, ds. \tag{5.8}$$

If $\delta l = 0$, the above curve C will be said to have a stationary length, and will be called a *geodesic*. It can be shown that a geodesic curve C satisfies the system of differential equations

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0, \tag{5.9}$$

where the functions Γ are defined by

$$\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\nu} \left(\frac{\partial g_{\nu\beta}}{\partial x^\gamma} + \frac{\partial g_{\gamma\nu}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\nu} \right); \tag{5.10}$$

these functions are called *Christoffel symbols*.

To deduce equations (5.9) let us first denote the expression under the radical in (5.8) by U for simplicity; then $U = 1$ for $\epsilon = 0$. The curve determined by a value of $\epsilon\,(\neq 0)$ can be considered to be obtained from the curve C, corresponding to $\epsilon = 0$, by making a variation of C such that each point of C undergoes a variation

$$\delta x^\alpha = \xi^\alpha(s) = \epsilon \left(\frac{\partial x^\alpha}{\partial \epsilon} \right)_{\epsilon=0};$$

the variation δx^α vanishes at the end points A and B of the curve C. For the first variation of the length l defined by (5.8) we have

$$\delta l = \epsilon \left(\frac{dl}{d\epsilon} \right)_{\epsilon=0} = \int_a^b \frac{\delta U}{2\sqrt{U}}\, ds,$$

or

$$\delta l = \frac{1}{2} \int_a^b \delta U\, ds, \tag{5.11}$$

since $U = 1$ for the curve C. Now

$$\frac{\partial U}{\partial \epsilon} = \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \frac{\partial x^\alpha}{\partial s} \frac{\partial x^\beta}{\partial s} \frac{\partial x^\gamma}{\partial \epsilon} + 2 g_{\gamma\delta} \frac{\partial x^\gamma}{\partial s} \frac{\partial^2 x^\delta}{\partial \epsilon\, \partial s};$$

hence

$$\delta U = \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} v^\alpha v^\beta \xi^\gamma + 2 g_{\alpha\beta} v^\alpha \frac{d\xi^\beta}{ds}, \tag{5.12}$$

where we have put

$$v^\alpha = \left(\frac{\partial x^\alpha}{\partial s}\right)_{\epsilon=0}.$$

Substituting (5.12) into (5.11) and integrating by parts, we obtain

$$\delta l = \int_a^b \left[\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} v^\alpha v^\beta - \frac{d}{ds}(g_{\gamma\delta} v^\delta)\right] \xi^\gamma \, ds,$$

where use is made of the fact that $\xi^\alpha(s)$ vanishes at the end points A and B. If $\delta l = 0$ for arbitrary infinitesimal displacements of the curve C, i.e. for arbitrary functions $\xi^\alpha(s)$, the bracket expression in the above integral must vanish along C; this leads immediately to the equations (5.9).

Among the totality of geodesic curves joining the points A and B of the space $\mathscr{V}$ there will evidently be one curve possessing the least length of any curve joining these points; in particular, if there is only one geodesic joining the points A and B, this will be the curve of shortest length. For example, if our space is of the nature of the surface of a cylinder, curves such as C and C' in Fig. 4 may be geodesics; in fact for this case there will be an infinity of such geodesics determined by encircling the cylinder 0, 1, 2, ... times, the geodesic C being the curve of shortest length between points A and B. There is an interesting property connected with any geodesic which can be stated roughly by saying that it represents the shortest curve between any two of its points provided that these points are sufficiently close together. More precisely we can say that for any point P of a geodesic C in the region $\mathscr{R}$ there exists a domain $\mathscr{S}$ of $\mathscr{R}$, such that the geodesic C is the curve of shortest length joining P to any of its points Q lying in the domain $\mathscr{S}$; this follows from the discussion in § 3.

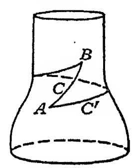

Fig. 4.

Any integral curve of (5.9) is such that along it the condition

$$g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = \text{const.} \tag{5.13}$$

is satisfied, where the constant in the right member of this equation is positive. To show this we have merely to differentiate the left member of (5.13) and then eliminate the second derivatives d^2x^α/ds^2 by means of (5.9); the expression so obtained then vanishes identically on account of (5.10). For a geodesic curve C, where s denotes the arc length measured from a fixed point A of C, the right member of (5.13) is of course equal to unity.

(*f*) *The affine connection.* The Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ defined by (5.10) can be considered as the components of a symmetric affine connection as described in § 2. Introducing this affine connection into the Riemann space, we are led to the natural identification of the paths in the sense of § 3 with the geodesics (5.9); geodesics thus appear as curves possessing the property of autoparallelism.

It is interesting to observe that the length of a vector remains unchanged under infinitesimal parallel displacement in Riemann space. For consider a curve C defined by $x^\alpha = \phi^\alpha(s)$ and a set of contravariant vectors with components $\xi^\alpha(s)$ parallel with respect to C. Differentiating, with respect to the parameter s, the expression which gives the square of the length of the vector at a point C, we have

$$\frac{d}{ds}(g_{\alpha\beta}\xi^\alpha\xi^\beta) = \left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} - g_{\mu\beta}\Gamma^\mu_{\alpha\gamma} - g_{\alpha\mu}\Gamma^\mu_{\beta\gamma}\right)\xi^\alpha\xi^\beta\frac{dx^\gamma}{ds},$$

use being made of equations of the type (2.3). But the expression in the parenthesis in the right member of the above equation vanishes identically on account of (5.10). A similar consideration applies to the covariant vector.

We have assumed in the above discussion that the quadratic differential form in the right member of (5.1) is positive definite. If this form is indefinite there will exist displacements with real components dx^α for which $ds^2 > 0$, $ds^2 = 0$, $ds^2 < 0$ will occur. Now it is evident that the above definitions and formulae can be carried over to the indefinite case or even the negative definite case provided first that we take account of the exceptional behaviour of the directions for which $ds^2 = 0$, and second make the formal modifications necessitated by those directions for which $ds^2 < 0$.* If the form (5.1) is indefinite and the constant in the right member of (5.13) is zero, the curve defined by (5.9) along which (5.13) is satisfied will be called a *geodesic of zero length*.†

A space whose metric is defined by (5.1) subject only to the condition that the determinant g does not vanish at any point of the region $\mathscr{R}$ will be called a *metric space*; in particular the word Riemann space will be reserved for a metric space for which the metric relationships are defined by a positive definite quadratic differential form. It is to be understood, furthermore, that the use of the term metric space implies that the affine connection is that determined by the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ defined by (5.10).

6. Space of distant parallelism

In the general affinely connected space it is of significance to say that a vector at a point P is parallel to a vector at an infinitely nearby point Q, e.g. the vector at Q parallel to the vector at P is determined from this latter vector by infinitesimal parallel displacement. It is, however, of no significance to say that a vector at a point P is parallel to a vector at a *distant* point Q of the region $\mathscr{R}$ covered by the coordinate system, but only that

* In this connection, L. P. Eisenhart has made use of the formal device of putting

$$ds^2 = e\, g_{\alpha\beta}\, dx^\alpha\, dx^\beta,$$

where e is plus or minus one, so that the right-hand member shall be positive, unless it is zero. See *Riemannian Geometry* (Princeton Univ. Press, 1926), p. 35.

† It should be observed that this is merely a "trade name" and that in fact the discussion of geodesic curves in sub-section (e) does not apply here.

these vectors are parallel with respect to a curve C joining the points P and Q. We shall now introduce into the affine space the possibility of distant parallelism, in an absolute sense. To do this we associate with each point of the underlying continuum a configuration consisting of n independent vectors which can be used to define a *local coordinate system*;* it is assumed that these vector configurations are in parallel orientation in such a way that the arbitrary orientation of the configuration at one point determines uniquely the orientation of the configurations at all points of the region $\mathscr{R}$. This affords the possibility of determining whether or not two vectors at different points are parallel, namely, by comparing their components in the local systems: *Two vectors are parallel if the corresponding components are equal when referred to a local system of coordinates*. If the vectors ξ and ζ at points P and Q of the region $\mathscr{R}$, as indicated in Fig. 5, are parallel, these vectors will be parallel in an absolute sense independently of the curve C, joining points P and Q.

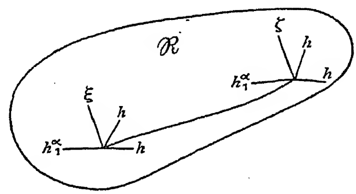

Fig. 5.

Let us denote by $h_i^\alpha(x)$ for $i = 1, \ldots, n$ the components of n contravariant vector fields; we observe that the Latin index i is thus used to denote the vector, and the Greek index α to denote the component of the vector. In general we shall adopt this convention throughout this section and related sections in the following work, i.e. we shall employ a Latin letter for an index which is of invariantive character with respect to the arbitrary transformations of the x^α coordinates, and a Greek letter in all contrary cases. Departures from this rule, as well as departures from the rule that an index which appears twice in a term is to be summed over the values of its range, will be such as are easily recognized on observation. By means of the quantities $h_i^\alpha(x)$ we impose on the underlying region $\mathscr{R}$ its structure as a space of distant parallelism in accordance with the following

Postulates of space structure

A. ***There exists a unique affine connection for the determination of the affine properties of the region $\mathscr{R}$.***

B. ***At each point P of the region $\mathscr{R}$ there is determined a configuration consisting of n independent vectors issuing from P.***

C. ***Corresponding vectors in the configurations, determined at two points P and Q of the region $\mathscr{R}$, are parallel.***

D. ***The components h_i^α of the vectors determining the configurations at any point P of the region $\mathscr{R}$ are analytic functions of the x^α coordinates.***

* The exact relation between the coordinates z^a of the local system and the coordinates x^α of underlying continuum will be established in § 30 of Chapter V.

The space whose structure is characterized by the above postulates will be spoken of as an *affine space of distant parallelism*. The n vectors with components $h_i^\alpha(x)$ which enter in Postulates B, C and D will be called the *fundamental vectors*. Since these vectors are independent by Postulate B, the determinant $|h_i^\alpha|$ will not vanish in the region $\mathscr{R}$. It is therefore possible to deduce from the fundamental vectors a system of n covariant vectors with components $h_\alpha^i(x)$ uniquely defined by the relations

$$h_i^\alpha h_\beta^i = \delta_\beta^\alpha, \quad h_k^\alpha h_\alpha^i = \delta_k^i; \tag{6.1}$$

the components h_α^i will be called the covariant components of the fundamental vectors to distinguish them from the contravariant components h_i^α of these vectors.

The condition that the fundamental vectors be parallel as demanded by Postulate C has its analytical expression in the equations

$$\frac{\partial h_i^\alpha}{\partial x^\gamma} + h_i^\beta \Delta_{\beta\gamma}^\alpha = 0, \tag{6.2}$$

where $\Delta_{\beta\gamma}^\alpha$ is used to denote the components of the affine connection. This gives

$$\Delta_{\beta\gamma}^\alpha = h_i^\alpha \frac{\partial h_\beta^i}{\partial x^\gamma} \tag{6.3}$$

as the equations which define the components $\Delta_{\beta\gamma}^\alpha$ of affine connection.

Let us now suppose the existence of another system of fundamental vectors in the same affine space of distant parallelism. Let us denote the contravariant components of these vectors by ${}^\star h_i^\alpha(x)$ in the x coordinate system, and let us put

$${}^\star h_i^\alpha = a_i^k h_k^\alpha; \tag{6.4}$$

these equations define the quantities a_i^k as analytic functions of the x^α coordinates in consequence of Postulate D. Multiplying both members of (6.4) by ${}^\star h_\beta^i h_\alpha^j$, we obtain the equations

$$h_\beta^j = a_k^j \, {}^\star h_\beta^k, \tag{6.5}$$

which represent the transformation induced by (6.4) on the covariant components h_α^i. On account of the uniqueness of determination of the components $\Delta_{\beta\gamma}^\alpha$ demanded by Postulate A, we must have

$$h_i^\alpha \frac{\partial h_\beta^i}{\partial x^\gamma} = {}^\star h_i^\alpha \frac{\partial \, {}^\star h_\beta^i}{\partial x^\gamma}.$$

From (6.4), (6.5), and these latter equations it readily follows that the quantities a_i^k are constants. It is in fact evident that the constants a_i^k are arbitrary subject to the condition that the determinant $|a_i^k|$ is not equal to zero, i.e. the components h_i^α in the affine space of distant parallelism are determined only to within a transformation (6.4) in which the coefficients a_i^k are arbitrary constants such that $|a_i^k|$ is not zero.

The possibility of metric determinations can be introduced into the above space of distant parallelism by replacing Postulate B by the two following postulates:

B_1. *There exists a unique quadratic differential form for the determination of the metric properties of the region* $\mathcal{R}$.

B_2. *At each point* P *of the region* $\mathcal{R}$ *there is determined a configuration consisting of* n *independent orthogonal unit vectors issuing from* P.

The space characterized by Postulates A, B_1, B_2, C and D is called a *metric space of distant parallelism*(9). The quadratic differential form in Postulate B will be used to determine the metric properties of the region $\mathcal{R}$ in accordance with the equation

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta,$$

as described in § 5. Since the fundamental vectors at a point P are orthogonal by Postulate B_2, the expression $g_{\alpha\beta} h_i^\alpha h_k^\beta$ is equal to zero for $i \neq k$; the condition that the fundamental vectors are unit vectors, likewise specified by Postulate B_2, means that for $i = k$ the expression $g_{\alpha\beta} h_i^\alpha h_k^\beta$ has the value ± 1. Hence we can write

$$g_{\alpha\beta} h_i^\alpha h_k^\beta = e_i \delta_k^i, \tag{6.6}$$

where the convenient notation e_i for $+1$ or -1 is introduced.* Since the fundamental vectors are independent by Postulate B_2, the determinant $| h_i^\alpha |$ does not vanish at any point P of the region $\mathcal{R}$;† hence the components h_α^i are defined by (6.1) and these can be used to solve the equations (6.6) for the quantities $g_{\alpha\beta}$. We thus obtain

$$g_{\alpha\beta} = \sum_{i=1}^{n} e_i h_\alpha^i h_\beta^i \tag{6.7}$$

as the equations which define the coefficients of the fundamental quadratic differential form. By (6.7) and Postulate D the $g_{\alpha\beta}$ are analytic functions of the coordinates x^α of the region $\mathcal{R}$. Also we see by taking the determinants of both members of (6.7) that the determinant $| g_{\alpha\beta} |$ does not vanish at any point P of $\mathcal{R}$.

In the metric space of distant parallelism the quantities a_i^k in (6.4) are not arbitrary constants since they must satisfy a condition of orthogonality, namely

$$\sum_{i=1}^{n} e_i a_k^i a_l^i = e_k \delta_l^k, \tag{6.8}$$

* This notation was introduced by L. P. Eisenhart, *loc. cit.* Chapter II.

† It may be observed in fact that the assumption of the independence of the fundamental vectors at a point P of $\mathcal{R}$ is not necessary as this can be deduced by the following consideration. If these vectors are dependent at a point P, i.e. if the determinant $|h_i^\alpha|$ is zero at P, there will exist constants c^i not all zero, such that $c^i h_i^\alpha = 0$ at P. Multiplying these equations by $g_{\alpha\beta} h_j^\beta$ and summing on the index α, it follows immediately from (6.6) that $c^i = 0$ contrary to the above hypothesis.

which is obtained from (6.5) and (6.7) as the direct result of the uniqueness of determination of the fundamental form specified by Postulate B_1.

Taking the determinant of both members of (6.8), we find that the determinant $|a^i_k|$ has the value ± 1. We can therefore define uniquely a set of quantities b^i_k by the equations

$$a^i_j b^j_k = \delta^i_k, \quad a^j_k b^i_j = \delta^i_k.$$

Another form of the conditions (6.8) which is sometimes useful can be derived in the following manner. Multiply both members of (6.8) by b^k_m so as to obtain

$$e_m a^m_l = e_l b^l_m,$$

or

$$a^m_l = e_l e_m b^l_m.$$

When we multiply both members of these latter equations through by $e_l a^k_l$ and sum on the index l, we find

$$\sum_{l=1}^{n} e_l a^k_l a^m_l = e_k \delta^k_n \tag{6.9}$$

The transformation (6.4) in which the coefficients a^k_i are constants satisfying (6.8) or (6.9) will be called an *orthogonal transformation* of the components of the fundamental vectors.

The components ξ^α of a contravariant vector in an affine or metric space of distant parallelism at a point P become

$$\xi^i = \xi^\alpha h^i_\alpha,$$

when referred to the vector configuration or local system at this point. To show that the components ξ^i of this vector are indeed unchanged by infinitesimal parallel displacement, we form the equations defining such displacements, namely

$$d\xi^\alpha = -\Delta^\alpha_{\beta\gamma}\xi^\beta dx^\gamma, \tag{6.10}$$

or

$$d\xi^\alpha + h^\alpha_j \frac{\partial h^j_\beta}{\partial x^\gamma}\xi^\beta dx^\gamma \equiv d\xi^\alpha + h^\alpha_j \xi^\beta dh^j_\beta = 0.$$

Multiplying these latter equations by h^i_α and summing on the index α, we obtain

$$h^i_\alpha d\xi^\alpha + \xi^\alpha dh^i_\alpha \equiv d(h^i_\alpha \xi^\alpha) \equiv d\xi^i = 0.$$

Thus the equations (6.10) reduce to equations expressing the vanishing of the quantities $d\xi^i$ which completes the proof. *It is moreover clear that while in general the components ξ^α of a contravariant vector will be changed by infinitesimal parallel displacement from a point P to a point Q along a curve C joining these points, the components ξ^α at the point Q will be independent of the route of displacement C.* Similar remarks of course likewise apply to the infinitesimal parallel displacements of covariant vectors.

The paths of a space of distant parallelism, i.e. the curves which are generated by continuously displacing a vector parallel to itself along its own direction, are those curves which are given as solutions of the system of equations

$$\frac{d^2x^\alpha}{ds^2}+\Lambda^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}=0, \tag{6.11}$$

where
$$\Lambda^\alpha_{\beta\gamma}=\tfrac{1}{2}(\Delta^\alpha_{\beta\gamma}+\Delta^\alpha_{\gamma\beta}).$$

The geodesics of a metric space of distant parallelism, on the other hand, are given as solutions of (5.9) in which the quantities $\Gamma^\alpha_{\beta\gamma}$ are defined by (5.10); hence in the space of distant parallelism the paths and geodesics are in general distinct.

7. Conformal space

A *conformal space* is one for which it is of significance only to compare lengths of line segments or vectors at a point P; in such a space length in an absolute sense is without meaning. To effect this comparison of lengths we assume the existence of a fundamental quadratic differential form with coefficients $g_{\alpha\beta}(x)$ of the same degree of generality as for the case of the metric space of § 5. Two vectors with components ξ^α and ζ^α associated with the same point P have lengths the squares of which are proportional to the values

$$(g_{\alpha\beta})_P\xi^\alpha\xi^\beta \quad\text{and}\quad (g_{\alpha\beta})_P\zeta^\alpha\zeta^\beta$$

respectively. Now, however, the quantities $g_{\alpha\beta}$ are determined only to within a factor of multiplication $\sigma(x)$, which we assume to be an analytic function of the coordinates x^α of the region $\mathscr{R}$. The transformation

$$g^\star_{\alpha\beta}=\sigma(x)\,g_{\alpha\beta} \tag{7.1}$$

which is thus imposed on the coefficients of the fundamental form, where $\sigma(x)$ is not zero in the region $\mathscr{R}$, is called a *conformal transformation*.

The angle θ between two directions d and δ as defined by (5.5) remains unchanged by the transformation (7.1); thus angle possesses an absolute significance in conformal space.

In general the paths or geodesics determined for a definite selection of the quantities $g_{\alpha\beta}$ by equations (5.9) are changed by the conformal transformation. *Geodesics of zero length are, however, unaltered by this transformation.* To see this we form the equations

$$\Lambda^{\star\alpha}_{\beta\gamma}=\Gamma^\alpha_{\beta\gamma}+\delta^\alpha_\beta\phi_\gamma+\delta^\alpha_\gamma\phi_\beta-g^{\alpha\epsilon}g_{\beta\gamma}\phi_\epsilon, \tag{7.2}$$

where
$$\phi_\beta=\frac{1}{2}\frac{\partial\log\sigma}{\partial x^\beta};$$

here $\Lambda^{\star\alpha}_{\beta\gamma}$ and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols determined by $g^\star_{\alpha\beta}$ and $g_{\alpha\beta}$ respectively.

Let C represent a geodesic of zero length before the application of the conformal transformation, i.e. C is a curve along which

$$\frac{d^2x^\alpha}{ds^2}+\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}=0 \tag{7.3}$$

and

(7.4) $$g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0$$

are satisfied. Eliminating $\Gamma^\alpha_{\beta\gamma}$ from (7.3) by means of (7.2) and omitting terms which vanish on account of (7.4), we have

(7.5) $$\frac{d^2x^\alpha}{ds^2}+\overset{\star}{\Lambda}{}^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}-\frac{\partial\log\sigma}{\partial x^\nu}\frac{dx^\nu}{ds}\frac{dx^\alpha}{ds}=0.$$

Upon making the change of parameter $s\rightarrow t$, where

$$\frac{dt}{ds}=\sigma(x),$$

we find that equations (7.5) become

$$\frac{d^2x^\alpha}{dt^2}+\overset{\star}{\Lambda}{}^\alpha_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt}=0,$$

and that (7.4) becomes

$$\overset{\star}{g}_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}=0.$$

In other words the curve C remains a geodesic of zero length after the conformal transformation (7.1).

Suppose that the projective properties of a conformal space are determined; this means that we must limit ourselves to those transformations (7.1) which leave geodesics invariant. The condition for this is that

(7.6) $$\delta^\alpha_\beta\phi_\gamma+\delta^\alpha_\gamma\phi_\beta-g^{\alpha\epsilon}g_{\beta\gamma}\phi_\epsilon=\delta^\alpha_\beta\psi_\gamma+\delta^\alpha_\gamma\psi_\beta,$$

where the ψ_β are any functions of the coordinates. If we put $\alpha=\beta$ in (7.6) and sum on these indices, we obtain

$$\phi_\gamma=\left(\frac{n+1}{n}\right)\psi_\gamma;$$

hence (7.6) becomes

(7.7) $$\delta^\alpha_\beta\psi_\gamma+\delta^\alpha_\gamma\psi_\beta-(n+1)\,g^{\alpha\epsilon}g_{\beta\gamma}\psi_\epsilon=0.$$

Now multiply (7.7) by $g_{\alpha\nu}g^{\beta\gamma}$ and sum on the repeated indices. We thus obtain a set of equations which reduce to

$$(n^2+n-2)\,\psi_\nu=0,$$

so that $\psi_\nu=0$ if $n\geqq 2$; hence the factor σ in (7.1) can at most be taken as a constant. Now it is evident that a conformal transformation (7.1) in which σ is a constant will leave unaltered all metric relationships existing throughout the region $\mathscr{R}$. We can therefore say that *the projective and conformal properties of a metric space determine its metric relationships uniquely.**

* In the theory of relativity, where the underlying space structure is assumed to be that of the metric space, this result has a direct and important interpretation. Since a freely moving mass particle moves along a geodesic in this theory, the motion of such particles determines the projective properties of the four-dimensional world. The conformal properties are determined by the *light cones*

$$g_{\alpha\beta}\,dx^\alpha dx^\beta=0,$$

which are invariant configurations under the conformal transformations (7.1) and which specify those world points in causal connection with one another. It follows from the above result that the metric relationships of the world are completely determined by the observation of the tracks of freely moving mass particles and the propagation of light. See H. Weyl, *loc. cit.* ref. (6).

8. Weyl space. Gauge

While the components of a vector are in general changed by an infinitesimal parallel displacement in a Riemann space, the length l of the vector remains unchanged as a result of this displacement. The observation of this fact was taken by H. Weyl as the basis of an extension of Riemann space which will be called a *Weyl space* (10). In analogy to the concept of infinitesimal parallel displacement let us introduce the concept of *congruent displacement of a vector* in consequence of which the length l of the vector is changed by an amount dl in accordance with the equation

$$dL = -L\phi_\alpha dx^\alpha \quad (L = l^2), \tag{8.1}$$

where the ϕ_α are analytic functions of the coordinates x^α of the region $\mathscr{R}$; these quantities ϕ_α arise as the coefficients of a homogeneous linear differential form $\phi_\alpha dx^\alpha$ which we shall refer to as the *fundamental linear differential form*. Roughly speaking, the Weyl space can be regarded as obtainable from the conformal space of § 7 by the introduction of this concept of congruent displacement; in this sense it is a space of conformal character. As the basis of precise discussion we shall lay down the following postulates by means of which the structure of the Weyl space is completely specified.

Postulates of space structure

A. *There exists a fundamental quadratic differential form of signature s; this form is determined to within a factor of multiplication $\lambda(x)$, called the gauge, and the quantity λ is a positive analytic function of the coordinates x^α of the region $\mathscr{R}$.*

B. *There exists a fundamental linear differential form which is uniquely determined when the gauge $\lambda(x)$ is fixed.*

C. *There exists a unique symmetric affine connection.*

D. *The coefficients ϕ_α and $g_{\alpha\beta}$ of the fundamental linear and quadratic differential forms respectively are analytic functions of the coordinates x^α of the region $\mathscr{R}$.*

Let us now examine certain of the consequences of the above postulates. Since the fundamental quadratic differential form is determined only to within a factor $\lambda(x)$ by Postulate A, we have the transformation equation

$$g^*_{\alpha\beta} = \lambda(x)\, g_{\alpha\beta}, \tag{8.2}$$

analogous to equation (7.1). The factor λ is always positive by Postulate A and this has as a consequence the fact that the signature s of the fundamental quadratic differential form will remain unchanged under transformations (8.2); this marks a slight refinement over the discussion of the conformal space in § 7 where the requirement that the factor $\sigma(x)$ be positive was not imposed. The region $\mathscr{R}$ will be said to be *calibrated* when the gauge $\lambda(x)$ is assigned.

Now suppose that the region $\mathscr{R}$ is definitely calibrated and that a vector at point P of $\mathscr{R}$ has length l on the basis of this calibration; by congruent displacement of this vector to an infinitely nearby point Q its length will be changed by a differential amount dl in accordance with (8.1) provided that dx^α gives the coordinate differences of the two points P and Q. If a new calibration of the region $\mathscr{R}$ is effected by means of (8.2), the length of this vector will become $l^\star$ in accordance with the equation

$$L^\star = \lambda L; \tag{8.3}$$

also

$$dL^\star = -L^\star \phi^\star_\alpha dx^\alpha \quad (L^\star = l^{\star 2}), \tag{8.4}$$

as a result of the above congruent displacement. From (8.1), (8.3) and (8.4) we therefore obtain

$$\phi^\star_\alpha = \phi_\alpha - \frac{1}{\lambda}\frac{\partial \lambda}{\partial x^\alpha}. \tag{8.5}$$

This equation gives the coefficients of the new fundamental linear differential form induced by the change of gauge, i.e. by the transformation (8.2).

The concepts of congruent displacement and of infinitesimal parallel displacement are inherent in the above postulates of space structure; more precisely these concepts are inherent in Postulates B and C respectively. We come now to the final postulate of the structure of the Weyl space which relates these two concepts of congruent and infinitesimal parallel displacement.

E. *A congruent displacement of a vector is likewise an infinitesimal parallel displacement of the vector, and conversely.*

On the basis of this last postulate we can calculate the components of affine connection in terms of the coefficients ϕ_α and $g_{\alpha\beta}$ of the two fundamental differential forms. By parallel displacement of a contravariant vector with components ξ^α from point P to an infinitely nearby point Q of the region $\mathscr{R}$, the components ξ^α change by a differential amount

$$d\xi^\alpha = -\Gamma^\alpha_{\beta\gamma}\xi^\beta dx^\gamma, \tag{8.6}$$

where the quantities $\Gamma^\alpha_{\beta\gamma}$ are symmetric in their lower indices; this follows from Postulate C. Since this displacement is also a congruent displacement by Postulate E, it follows that the square of the length of the vector, namely

$$L = g_{\alpha\beta}\xi^\alpha\xi^\beta, \tag{8.7}$$

changes in accordance with equation (8.1). Hence

$$2g_{\alpha\beta}\xi^\alpha d\xi^\beta + \xi^\alpha\xi^\beta dg_{\alpha\beta} = -(g_{\alpha\beta}\xi^\alpha\xi^\beta)\phi_\gamma dx^\gamma,$$

from (8.1) and (8.7). Elimination of the $d\xi^\beta$ in these last equations by means of (8.6) leads to the equations

$$g_{\alpha\nu}\Gamma^\nu_{\beta\gamma}+g_{\beta\nu}\Gamma^\nu_{\alpha\gamma}=\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}+g_{\alpha\beta}\phi_\gamma; \tag{8.8}$$

from these equations, we obtain

$$g_{\beta\nu}\Gamma^\nu_{\gamma\alpha}+g_{\gamma\nu}\Gamma^\nu_{\beta\alpha}=\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}+g_{\beta\gamma}\phi_\alpha, \tag{8.9}$$

$$g_{\gamma\nu}\Gamma^\nu_{\alpha\beta}+g_{\alpha\nu}\Gamma^\nu_{\gamma\beta}=\frac{\partial g_{\gamma\alpha}}{\partial x^\beta}+g_{\gamma\alpha}\phi_\beta, \tag{8.10}$$

by cyclic interchange of indices. Now add equations (8.8) and (8.10) and from the equations so obtained subtract (8.9); there result the equations

$$g_{\alpha\nu}\Gamma^\nu_{\beta\gamma}=\frac{1}{2}\left[\left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}+\frac{\partial g_{\gamma\alpha}}{\partial x^\beta}-\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)+(g_{\alpha\beta}\phi_\gamma+g_{\gamma\alpha}\phi_\beta-g_{\beta\gamma}\phi_\alpha)\right],$$

or

$$\Gamma^\alpha_{\beta\gamma}=\tfrac{1}{2}g^{\alpha\nu}\left(\frac{\partial g_{\nu\beta}}{\partial x^\gamma}+\frac{\partial g_{\gamma\nu}}{\partial x^\beta}-\frac{\partial g_{\beta\gamma}}{\partial x^\nu}\right)+\tfrac{1}{2}(\delta^\alpha_\beta\phi_\gamma+\delta^\alpha_\gamma\phi_\beta-g^{\alpha\nu}g_{\beta\gamma}\phi_\nu). \tag{8.11}$$

These latter equations define the components of affine connection in the Weyl space.

To sum up: let us observe that in a Weyl space, *first* we can only speak of length in an absolute sense after a definite calibration of the region $\mathcal{R}$ has been effected, the square of the length of a contravariant vector with components ξ^α at a point P of $\mathcal{R}$ then being given by the equation (8.7); *second*, the coefficients $g_{\alpha\beta}$ of the fundamental quadratic differential form are subject to transformations of the type (8.2) and that this induces a transformation of the type (8.5) on the coefficients ϕ_α of the fundamental linear differential form; *third*, the equation (8.1) gives the differential change dl in the length l of a contravariant vector with components ξ^α at a point P when this vector is displaced by congruent or infinitesimal parallel displacement from point P to the infinitely nearby point Q of the region $\mathcal{R}$, the quantities dx^α representing the coordinate differences of these points; and *finally* under these same conditions the change $d\xi^\alpha$ in the components of the vector due to infinitesimal parallel displacement are given by (8.6) in which the components of affine connection are defined by (8.11).

When the region $\mathcal{R}$ has been definitely calibrated, measurements of lengths, angles, volumes, etc., can be made in the Weyl space exactly as in the metric space discussed in § 5; in this connection we observe that the measurement of the angle between two vectors at a point P of the region $\mathcal{R}$ is defined independently of the ambiguity of the type (8.2) in the coefficients of the fundamental quadratic differential form. It is an interesting fact that this ambiguity in the coefficients $g_{\alpha\beta}$, i.e. more precisely, the simultaneous transformations (8.2) and (8.5) leave unchanged the components $\Gamma^\alpha_{\beta\gamma}$ of the

affine connection; this can be established from (8.11) by direct calculation. Hence the paths of the Weyl space have a significance independent of the transformations (8.2) and (8.5); this is also true of the geodesics of zero length of this space since the discussion in § 7 shows that such geodesics are invariant under the transformation (8.2).

In § 3 we defined a system of *normal coordinates* with origin at an arbitrary point P of the region $\mathscr{R}$ and this system of coordinates possesses the property that with respect to it the components of affine connection vanish at the origin. This results in the consequence that the change $d\xi^\alpha$ vanishes for infinitesimal parallel displacements from the origin of a normal coordinate system to any infinitely nearby point Q. An analogous result regarding congruent displacements can be established—it is possible to calibrate the region $\mathscr{R}$ so that for an arbitrary point P of $\mathscr{R}$ the length of any vector with components ξ^α will be unchanged when moved from P to an infinitely nearby point Q by a congruent displacement. To accomplish this we have in fact merely to choose the gauge $\lambda(x)$ so that

$$\phi_\alpha = \frac{\partial \log \lambda}{\partial x^\alpha}$$

at the point P; then the right member of (8.5) will vanish at the point P and the above statement is proved.

9. Transformation theory of space

It is clear that the system of coordinates x^α of the region $\mathscr{R}$ used in the preceding sections can be replaced by any system of coordinates $\bar{x}^\alpha$ defined by a transformation of the form (1.2) in which the f^α are analytic functions throughout the region $\mathscr{R}$, and that these latter coordinates $\bar{x}^\alpha$ will do equally well as the basis of our discussion. We shall express the principle of the equivalence of all such systems of coordinates by the following statement serving to emphasize the importance of this concept.

Geometrical Principle of Relativity. In the analytical treatment of the problems of space, any system of coordinates x^α of the region $\mathscr{R}$ can be replaced by a system of coordinates $\bar{x}^\alpha$, related to the coordinates x^α by a transformation

$$x^\alpha = f^\alpha(\bar{x}^1, \ldots, \bar{x}^n), \tag{9.1}$$

in which the f^α are analytic functions of the variables $\bar{x}^1, \ldots, \bar{x}^n$ throughout the region $\mathscr{R}$.

This principle implies that the analytic form of our spatial or geometrical relationships must be invariant under the group of analytic transformations of the coordinates x^α of the region $\mathscr{R}$. Let us apply this test to certain of the relationships of the preceding sections. For example, consider what happens to the equations (2.1) defining an infinitesimal parallel displacement of a vector, under an analytic transformation (9.1). The components ξ^α and the differentials dx^β enjoy the contravariant vector transformations

$$(a)\ \xi^\alpha = \frac{\partial x^\alpha}{\partial \bar{x}^\sigma}\bar{\xi}^\sigma \quad \text{and} \quad (b)\ dx^\beta = \frac{\partial x^\beta}{\partial \bar{x}^\tau} d\bar{x}^\tau \tag{9.2}$$

respectively. By differentiation of (9.2 (a)) we have

$$d\xi^\alpha = \frac{\partial x^\alpha}{\partial \bar{x}^\sigma} d\bar{\xi}^\sigma + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\sigma \partial \bar{x}^\tau} \bar{\xi}^\sigma d\bar{x}^\tau;$$

hence we obtain

$$(9.3) \qquad d\xi^\alpha + L^\alpha_{\beta\gamma} \xi^\beta dx^\gamma = \frac{\partial x^\alpha}{\partial \bar{x}^\epsilon} \left[d\bar{\xi}^\epsilon + \frac{\partial \bar{x}^\epsilon}{\partial x^\nu} \left(\frac{\partial^2 x^\nu}{\partial \bar{x}^\sigma \partial \bar{x}^\tau} + L^\nu_{\beta\gamma} \frac{\partial x^\beta}{\partial \bar{x}^\sigma} \frac{\partial x^\gamma}{\partial \bar{x}^\tau} \right) \bar{\xi}^\sigma d\bar{x}^\tau \right].$$

The condition that (2.1) be invariant in form, i.e. that

$$d\bar{\xi}^\epsilon = -\bar{L}^\epsilon_{\sigma\tau} \bar{\xi}^\sigma d\bar{x}^\tau,$$

is therefore that

$$(9.4) \qquad \bar{L}^\epsilon_{\sigma\tau} = \frac{\partial \bar{x}^\epsilon}{\partial x^\nu} \left(\frac{\partial^2 x^\nu}{\partial \bar{x}^\sigma \partial \bar{x}^\tau} + L^\nu_{\beta\gamma} \frac{\partial x^\beta}{\partial \bar{x}^\sigma} \frac{\partial x^\gamma}{\partial \bar{x}^\tau} \right).$$

These constitute the equations of transformation of the components $L^\alpha_{\beta\gamma}$ demanded by the above principle; evidently we could likewise arrive at the transformation (9.4) by a similar consideration of (2.2).

In particular (9.4) gives

$$(9.5) \qquad \bar{\Gamma}^\alpha_{\beta\gamma} = \frac{\partial \bar{x}^\alpha}{\partial x^\nu} \left(\frac{\partial^2 x^\nu}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} + \Gamma^\nu_{\lambda\mu} \frac{\partial x^\lambda}{\partial \bar{x}^\beta} \frac{\partial x^\mu}{\partial \bar{x}^\gamma} \right)$$

and

$$(9.6) \qquad \bar{\Omega}^\alpha_{\beta\gamma} = \Omega^\lambda_{\mu\nu} \frac{\partial \bar{x}^\alpha}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \bar{x}^\beta} \frac{\partial x^\nu}{\partial \bar{x}^\gamma}$$

as the equations of transformation of the symmetric and skew-symmetric parts of the components $L^\alpha_{\beta\gamma}$. The transformation (9.5) likewise results from the requirement of invariance of the equations of the paths (3.1). Evidently the transformation (9.5) also applies when the $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols in a metric space; these equations likewise furnish the equations of transformation of the Christoffel symbols in a conformal or Weyl space when (7.1) or (8.2) is taken as the identical transformation.

In addition to the invariance of form of the analytical equations defining geometrical configurations as illustrated by the above examples, the above principle of relativity requires likewise an invariance of value of the geometrical magnitudes associated with these configurations. To illustrate this, take the case of the equation (5.1) defining the distance ds between two points P and Q with coordinates x^α and $x^\alpha + dx^\alpha$ respectively. The requirement of invariance of ds under transformations (9.1) gives

$$g_{\alpha\beta} dx^\alpha dx^\beta = \bar{g}_{\sigma\tau} d\bar{x}^\sigma d\bar{x}^\tau;$$

hence

$$g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\sigma} \frac{\partial x^\beta}{\partial \bar{x}^\tau} d\bar{x}^\sigma d\bar{x}^\tau = \bar{g}_{\sigma\tau} d\bar{x}^\sigma d\bar{x}^\tau,$$

making the substitution (9.2 (b)), and since this equation must hold for arbitrary values $d\bar{x}^\alpha$, we obtain

$$(9.7) \qquad = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\sigma} \frac{\partial x^\beta}{\partial \bar{x}^\tau}.$$

These equations give the transformation of the coefficients of the fundamental quadratic differential form induced by the coordinate transformation (9.1).

Similarly consider the equation (5.5) which defines the angle θ between two displacements d and δ—here the configuration in question consists of the displacements d and δ and the angle θ is the associated magnitude. Use of (9.2 (b)) and (9.7) leads immediately to the invariance of the right member of (5.5) and hence to the invariance of value of the angle θ under coordinate transformations (9.1).

It is obvious that invariance under coordinate transformations (9.1), as illustrated by the above examples, is a common property of geometrical configurations (defined analytically by equations) and the magnitudes associated with them. Indeed this property of invariance is but the necessary result of the fact that one system of coordinates is taken to be on an equal footing with any other in the analytical treatment of geometrical problems, i.e. the geometrical principle of relativity. In addition, however, to invariance under coordinate transformations we must demand in the case of conformal geometry an invariance under the transformation (7.4) and in the case of the Weyl geometry a similar invariance with respect to transformations (8.2) and (8.5). We are hereby provided with a means by which we can test any equation or expression to see if it can possibly represent a true geometrical entity—invariance of the above type being the necessary requirement. This suggests that we assume a point of view in which this requirement of invariance is taken to be fundamental; we shall refer to this as the *invariant-theoretic view point.*

On the basis of this point of view we can construct our invariant entities first and then inquire later as to their geometrical interpretation. For example, we could have been led to the equations of the path (3.1) on account of the invariance of form of these equations under coordinate transformations. Also we could have arrived at the right member of (5.5) as a very simple invariant associated with two displacements d and δ in a metric, conformal or Weyl space. Similar remarks can be made with regard to other geometrical configurations and their associated magnitudes.

The invariant-theoretic viewpoint will be adopted in the following work. Indeed our object will be the construction of invariants in a rather general sense and the geometrical interpretation of these invariants will be pushed into the background. We shall in fact be concerned exclusively with the analysis of space *as such* and not at all with the treatment of the configurations, e.g. curves, surfaces, etc., which can exist in space.

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(5) The equations of the geodesics for a two-dimensional surface were written in a form which is independent of change of parameter by G. Darboux, *Leçons sur la Théorie Générale des Surfaces*, 2nd ed. (Gauthier-Villars, 1915), **2**, p. 417. The corresponding form (3.4) for the equations of the paths was suggested by J. L. Synge in a letter to O. Veblen.

(6) The discovery that there can be more than one affine connection for the same system of paths is due to H. Weyl, "Zur Infinitesimalgeometrie: Einordung der projektiven und der konformen Auffassung", *Göttingen Nachrichten* (1921), pp. 99–112. See also L. P. Eisenhart, "Spaces with corresponding paths", *Proc. N.A.S.* **8** (1922), pp. 233–8; O. Veblen, "Projective and affine geometry of paths", *Proc. N.A.S.* **8** (1922), pp. 347–50; O. Veblen and T. Y. Thomas, "The geometry of paths", *Trans. Amer. Math. Soc.* **25** (1923), pp. 551–608.

(7) T. Y. Thomas, "On the projective and equi-projective geometries of paths", *Proc. N.A.S.* **11** (1925), pp. 199–203.

(8) See B. Riemann, ref. (4).

(9) The affine connection of a space of distant parallelism was used in a process of covariant differentiation from a purely analytical point of view by R. Weitzenböck, "Neuere Arbeiten der algebraischen Invariantentheorie, Differentialinvarianten", *Enzyklopädie*, Bd. III, 3. Teil (1921); also *Invariantentheorie* (Groningen: Noordhoff, 1923), p. 319. É. Cartan seems to be the first to have shown that if the curvature tensor vanishes the change in the components of a vector due to infinitesimal parallel displacement is independent of the route of displacement. "Sur les variétés à connexion affine et la théorie de la relativité généralisée", *Ann. l'éc. norm. sup.* **40** (1923), pp. 325–412. Also G. Vitali noted that the parallelism determined by the Weitzenböck connection is independent of the curve with respect to which it is considered, and that parallel directions at different points make the same angles with the n congruences defined by the h_i^α when the metric defined by (6.7), with $e_i = 1$, is introduced. See "Una derivazione covariante formata coll' ausilio di n sistemi covarianti del 1 ordine",

Atti d. Soc. Ligustica di Sc. e Lett. **2** (1924), pp. 248–53; "Intorno ad una derivazione nel calcolo assoluto", *ibid.* **4** (1925), pp. 287–91. See also L. P. Eisenhart, "Linear connections of a space which are determined by simply transitive continuous groups", *Proc. N.A.S.* **11** (1925), pp. 246–50; "Non-Riemannian geometry", *Trans. Amer. Math. Soc.* (1927), p. 48; E. Bortollotti, "On metric connections with absolute parallelism", *Proc. Kon. Akad. Amsterdam,* **30** (1927), pp. 216–18. The metric spaces of distant parallelism as presented in § 6 were employed in a Unified Field Theory by A. Einstein: "Riemann-Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelismus", *Sitzungsberichte Preuss. Akad.* (1928), pp. 217–21; "Neue Möglichkeit für eine einheitliche Feldtheorie von Gravitation und Elektrizität", *ibid.* pp. 224–7. Also the affine space of distant parallelism was considered by H. P. Robertson, "Groups of motions in spaces admitting absolute parallelism", *Ann. of Math.* (2), **33** (1932), pp. 496–520.

(10) H. Weyl, "Gravitation und Elektrizität", *Sitzungsberichte Preuss. Akad.* (1918), pp. 465–80. See also H. Weyl, *Raum, Zeit, Materie,* 5th ed. (Julius Springer, 1921).

CHAPTER II

AFFINE AND RELATED INVARIANTS

10. Tensors

A GENERALIZATION of the vector called the *tensor* is fundamental in the development of the programme mentioned at the end of the preceding chapter; we proceed to define this concept of tensor[1].

Definition of Affine Tensor. A set of functions $T^{\alpha\ldots\beta}_{\gamma\ldots\delta}(x)$ *of the coordinates* x^α *of the region* $\mathscr{R}$, *where* $\alpha,\ldots,\beta,\gamma,\ldots,\delta = 1,\ldots,n$, *constitute the components of a relative affine tensor* T *of weight* W *with respect to the* x *coordinate system, provided that the functions* $T^{\alpha\ldots\beta}_{\gamma\ldots\delta}$ *transform according to the equations*

$$\overline{T}^{\mu\ldots\nu}_{\sigma\ldots\tau}(\overline{x}) = (x\overline{x})^W\, T^{\alpha\ldots\beta}_{\gamma\ldots\delta}(x)\frac{\partial \overline{x}^\mu}{\partial x^\alpha}\cdots\frac{\partial \overline{x}^\nu}{\partial x^\beta}\frac{\partial x^\gamma}{\partial \overline{x}^\sigma}\cdots\frac{\partial x^\delta}{\partial \overline{x}^\tau}, \tag{10.1}$$

where

$$(x\overline{x}) = \begin{vmatrix} \dfrac{\partial x^1}{\partial \overline{x}^1} & \cdots & \dfrac{\partial x^1}{\partial \overline{x}^n} \\ \vdots & & \vdots \\ \dfrac{\partial x^n}{\partial \overline{x}^1} & \cdots & \dfrac{\partial x^n}{\partial \overline{x}^n} \end{vmatrix}$$

when the coordinates x^α *are transformed by* (1.2). *The object obtained by abstraction from the above components with respect to the totality of coordinate systems whose coordinates are related by* (1.2) *is called the tensor* T.

It is to be considered that the components $T^{\alpha\ldots\beta}_{\gamma\ldots\delta}(x)$ which are associated with a point P of the region $\mathscr{R}$ are transformed by (10.1) into the components $\overline{T}^{\mu\ldots\nu}_{\sigma\ldots\tau}(\overline{x})$ which are associated with the same point P.

A tensor T in the sense of the above definition is sometimes spoken of as a *tensor field*. We shall also adopt this terminology on occasion when we wish to emphasize the character of the components of T as functions of the coordinates x^α, otherwise the more abbreviated terminology of tensor will be employed.

It is clear that if a set of quantities $T^{\alpha\ldots\beta}_{\gamma\ldots\delta}$, where the indices $\alpha,\ldots,\delta$ take on values $1,\ldots,n$, are assigned as arbitrary functions of the coordinates x^α of the region $\mathscr{R}$, then these functions together with the law of transformation (10.1) will lead to the definition of a tensor T in the sense of the above definition.

The law of transformation (10.1) of the components of a tensor T is *transitive*. Thus if we denote the components of a tensor T by A for brevity and if we suppose (1) that $A \to \overline{A}$ as the result of a coordinate transformation G_1 and (2) that $\overline{A} \to \tilde{A}$ as the result of a coordinate transformation G_2, then

$A \to \tilde{A}$ as the result of the transformation G_3 which is the resultant of the two transformations G_1 and G_2.

If the components of a tensor T are of the form $T^{\alpha\dots\beta}$ the tensor is called a *contravariant tensor*, and if the components have the form $T_{\gamma\dots\delta}$ it is called a *covariant tensor*; correspondingly the indices $\alpha, \dots, \beta$ are called *contravariant indices* and the indices $\gamma, \dots, \delta$ are called *covariant indices*. Thus the contravariant and covariant tensor is a direct generalization of the contravariant and covariant vector, respectively. Since indices of both contravariant and covariant type appear on the components $T^{\alpha\dots\beta}_{\gamma\dots\delta}$ of the tensor T in the above definition, this tensor is sometimes called a *mixed tensor*.

The number of indices which appear in the symbol for the components of a tensor is called the *rank* of the tensor. Thus $T_{\mu\nu}$ represents the components of a covariant tensor of rank 2, and $T^{\sigma\tau}_{\lambda\mu\nu}$ represents the components of a mixed tensor of rank 5. A tensor of rank 0 is a scalar and a tensor of rank 1 is a covariant or contravariant vector.

The following is a list of some of the simpler operations that can be performed on tensors, all of which can be proved directly by recourse to the law of transformation of the components of the tensor.

(*a*) *Addition*. The components $A^{\alpha\dots\beta}_{\gamma\dots\delta}$ and $B^{\alpha\dots\beta}_{\gamma\dots\delta}$ of two tensors of the same weight can be added to produce the components of a single tensor. Thus

$$C^{\alpha\dots\beta}_{\gamma\dots\delta} = A^{\alpha\dots\beta}_{\gamma\dots\delta} + B^{\alpha\dots\beta}_{\gamma\dots\delta} \tag{10.2}$$

are the components of a tensor. Similarly the differences between the corresponding components of two tensors constitute the components of a single tensor.

(*b*) *Multiplication*. If we multiply together all the components $A^{\alpha\dots\beta}_{\gamma\dots\delta}$ of a tensor by all the components $B^{\mu\dots\nu}_{\xi\dots\pi}$ of another tensor, we obtain a set of quantities

$$C^{\alpha\dots\beta\,\mu\dots\nu}_{\gamma\dots\delta\,\xi\dots\pi} = A^{\alpha\dots\beta}_{\gamma\dots\delta}\, B^{\mu\dots\nu}_{\xi\dots\pi} \tag{10.3}$$

which constitute the components of a new tensor.

(*c*) *Contraction*. The components $A^{\alpha\beta\dots\gamma}_{\lambda\mu\dots\nu}$ of a tensor can be used to define a set of quantities

$$B^{\beta\dots\gamma}_{\mu\dots\nu} = A^{\alpha\beta\dots\gamma}_{\alpha\mu\dots\nu} \tag{10.4}$$

which possess a tensor character. The components $B^{\beta\dots\gamma}_{\mu\dots\nu}$ are said to be obtained from the components $A^{\alpha\beta\dots\gamma}_{\lambda\mu\dots\nu}$ by contracting the indices α and λ. It is evident that any index of the set $\alpha, \beta, \dots, \gamma$ and any index of the set $\lambda, \mu, \dots, \nu$ can be contracted in this manner. A tensor is said to be *symmetric* with respect to two contravariant or covariant indices α, β which appear in the symbol of its components, if the interchange of α and β leaves unaltered the value of all components of the tensor. It is *skew-symmetric* if the interchange

of α and β changes the sign of all components of the tensor. For example the tensor with components $A^{\alpha\beta\gamma}$ is symmetric in the indices α and β if

$$A^{\alpha\beta\gamma}=A^{\beta\alpha\gamma},$$

and it is skew-symmetric if

$$A^{\alpha\beta\gamma}=-A^{\beta\alpha\gamma}.$$

It is readily verified that the symmetric or skew-symmetric properties of the components persist under transformations of coordinates. A tensor which is symmetric or skew-symmetric with respect to all pairs of adjacent contravariant or covariant indices in the symbol of its components is spoken of simply as a symmetric or skew-symmetric tensor.

If V_α are the components of a covariant vector V, then the quantities $W_{\alpha\beta}$, defined by

$$W_{\alpha\beta}=\frac{\partial V_\alpha}{\partial x^\beta}-\frac{\partial V_\beta}{\partial x^\alpha},$$

constitute the components of a skew-symmetric tensor W; this tensor is called the *curl of the vector* V.

Let us say that two tensors A and B are of the same *kind* if (1) the symbol of the components of A contains the same number of contravariant indices and likewise the same number of covariant indices as the symbol of the components of the tensor B and (2) the tensors A and B are of the same weight W. Two tensors of the same kind are therefore of the same rank, but it is not true that if the ranks of two tensors are the same, these tensors are necessarily of the same kind.

Definition of Equality. Two tensors A and B of the same kind will be said to be equal if their corresponding components are equal in all coordinate systems.

THEOREM. *If the components of a tensor A are equal to the corresponding components of a tensor B, of the same kind as A, in one coordinate system, then A and B are equal.*

To prove this theorem we have merely to construct the equations

$$\bar{A}^{\lambda\ldots\mu}_{\nu\ldots\xi}-\bar{B}^{\lambda\ldots\mu}_{\nu\ldots\xi}=(A^{\alpha\ldots\beta}_{\gamma\ldots\delta}-B^{\alpha\ldots\beta}_{\gamma\ldots\delta})\frac{\partial\bar{x}^\lambda}{\partial x^\alpha}\cdots\frac{\partial\bar{x}^\mu}{\partial x^\beta}\frac{\partial x^\gamma}{\partial\bar{x}^\nu}\cdots\frac{\partial x^\delta}{\partial\bar{x}^\xi}, \tag{10.5}$$

and observe that, by hypothesis

$$A^{\alpha\ldots\beta}_{\gamma\ldots\delta}=B^{\alpha\ldots\beta}_{\gamma\ldots\delta}.$$

COROLLARY. *If the components of a tensor vanish in one coordinate system they vanish in every coordinate system.*

The importance of the concept of a tensor is largely due to this last corollary, since we are thereby enabled to express our equations in a form which will remain valid under arbitrary transformations of coordinates; on this account too the use of the tensor is admirably adapted to a discussion of spatial problems, on the basis of the invariant-theoretic viewpoint described at the

end of Chapter I. If all the components of a tensor T vanish, we shall say that the tensor T vanishes as a matter of convenient terminology.

A theorem of the general nature of the above theorem and which is of possible although restricted application can also be stated(2). Preliminary to this we lay down the following

Definition of the Class of a Component. The components $T^{a_1\dots a_k}_{a_{k+1}\dots a_l}$ *and* $T^{b_1\dots b_k}_{b_{k+1}\dots b_l}$ *of the tensor* T *will be said to be in the same class if when* $a_i = a_j$ *we have* $b_i = b_j$ *and when* $a_i \neq a_j$ *we have* $b_i \neq b_j$.

THEOREM. *If one component* α *of a tensor* A *is equal to the corresponding component* β *of a tensor* B, *of the same kind as* A, *in all coordinate systems, then the corresponding components of the classes of* α *and* β *are equal in all coordinate systems.*

By hypothesis we have

$$\bar{A}^{a\dots b}_{c\dots d} = \bar{B}^{a\dots b}_{c\dots d}, \tag{10.6}$$

where $a, \dots, b, c, \dots, d = 1, \dots, n$ have particular values; this relation holds in all coordinate systems. Now let $\bar{\alpha}, \dots, \bar{\beta}, \bar{\gamma}, \dots, \bar{\delta}$ denote particular values of $\alpha, \dots, \beta, \gamma, \dots, \delta$ in (10.2) such that $A^{\bar{\alpha}\dots\bar{\beta}}_{\bar{\gamma}\dots\bar{\delta}}$ is in the same class as $A^{a\dots b}_{c\dots d}$. We wish to show that

$$A^{\bar{\alpha}\dots\bar{\beta}}_{\bar{\gamma}\dots\bar{\delta}} = B^{\bar{\alpha}\dots\bar{\beta}}_{\bar{\gamma}\dots\bar{\delta}};$$

this however follows from the fact (1) that the left member of (10.5) is zero by (10.6) and (2) that we can make a coordinate transformation such that the coefficient of

$$A^{\bar{\alpha}\dots\bar{\beta}}_{\bar{\gamma}\dots\bar{\delta}} - B^{\bar{\alpha}\dots\bar{\beta}}_{\bar{\gamma}\dots\bar{\delta}}$$

is not equal to zero, while all other such coefficients in the right member of (10.5) are equal to zero. A transformation of this sort is

$$\begin{aligned}
\bar{x}^a &= x^{\bar{\alpha}}, \\
&\dots\dots\dots \\
\bar{x}^b &= x^{\bar{\beta}}, \\
\bar{x}^c &= x^{\bar{\gamma}}, \\
&\dots\dots\dots \\
\bar{x}^d &= x^{\bar{\delta}}, \\
\bar{x}^e &= \phi^e(x^1, \dots, x^n), \quad e \neq a, \dots, b, c, \dots, d.
\end{aligned}$$

COROLLARY (*a*). *If one component of each class of a tensor* A *is equal to the corresponding component of a tensor* B, *of the same kind as* A, *in all coordinate systems, then* A *and* B *are equal.*

COROLLARY (*b*). *If one component of a vector is equal to the corresponding component of another vector of the same kind in all coordinate systems, then the two vectors are equal.*

All the above remarks have had reference to relative affine tensors of weight W although only the abbreviated designation of tensor was employed. If the weight W is 0 the tensor is called an *absolute tensor* and if the weight W is equal to 1 it is called a *tensor density*, the symbol for its components then being usually written $\mathfrak{T}^{\alpha\dots\beta}_{\gamma\dots\delta}$. The justification of the name tensor density lies in the fact that the law of transformation of the integral

$$\int\int\dots\int \mathfrak{T}^{\alpha\dots\beta}_{\gamma\dots\delta}\, dV,$$

extended over a definite n-dimensional domain $\mathcal{D}$ of $\mathcal{R}$, approaches more and more closely the law of transformation of an absolute tensor with components $T^{\alpha\dots\beta}_{\gamma\dots\delta}$ as the domain of integration $\mathcal{D}$ closes down on a point. Thus

the tensor density represents a sort of tensor of weight 0 per unit of coordinate volume. As, however, we shall usually be concerned with absolute tensors in the following developments, we shall use the word tensor in that sense and only speak of a relative tensor of weight W when the weight W is different from 0.

Equations (9.7) show that the quantities $g_{\alpha\beta}$ constitute the components of a covariant tensor of rank 2. This is an important tensor called the *fundamental metric tensor*. Moreover the quantities $g^{\alpha\beta}$ defined by (5.4) can easily be shown to form the components of a contravariant tensor; we refer to this tensor as the contravariant form of the fundamental metric tensor.

In a metric space, a metric space of distant parallelism, and a Weyl space, the gauge being fixed, we can raise or lower the indices in the symbol of the components of a tensor by an operation involving the components of the fundamental metric tensor. For example we can deduce from the components V_α of a covariant vector the components

$$V^\beta = g^{\alpha\beta} V_\alpha$$

of a contravariant vector; similarly the components

$$T_\gamma^\beta = g_{\alpha\gamma} T^{\alpha\beta}$$

can be deduced from the components of a contravariant tensor of the second rank. A tensor derived in this manner from a tensor T is called an *associated tensor* of T. It is likewise possible to modify the weight of a tensor T by an operation having its origin in the transformation equation

$$\bar{g} = (x\bar{x})^2 g, \tag{10.7}$$

deducible from (9.7). By (10.7) the determinant g is the component of a relative scalar of weight 2. Hence if T is a relative tensor of weight W, the quantities

$$g^{V/2} T^{\alpha \dots \beta}_{\gamma \dots \delta}$$

constitute the components of a relative tensor of weight $V + W$; in particular, for example, if V^α are the components of a contravariant vector, i.e. a relative vector of weight zero, then

$$\mathfrak{V}^\alpha = \sqrt{g}\, V^\alpha$$

will be the components of contravariant vector density.

An analogous process in a space of distant parallelism enables us to construct *a set of relative scalars of weight* W from a relative tensor T of the same weight. Thus we have

$$U^{i \dots j}_{k \dots l} = T^{\alpha \dots \beta}_{\gamma \dots \delta}\, h^i_\alpha \dots h^j_\beta\, h^\gamma_k \dots h^\delta_l \tag{10.8}$$

as the components of these scalars. Under a transformation (6.5) of the fundamental vectors, the components $U^{i \dots j}_{k \dots l}$ transform by the equations

$$\star U^{p \dots q}_{r \dots s} a^i_p \dots a^j_q = U^{i \dots j}_{k \dots l} a^k_r \dots a^l_s\,; \tag{10.9}$$

we call these scalar components the components of the tensor T with respect to the local coordinate systems defined by the fundamental vector configurations of the space of distant parallelism.

If $\xi^\alpha(x)$ denote the components of a contravariant vector where the ξ^α are analytic functions of the coordinates x^α of the region $\mathscr{R}$, we can find a coordinate system for a limited domain of $\mathscr{R}$ with respect to which the components of this vector are given by δ^α_1. Thus consider the equation

$$\xi^\alpha \frac{\partial f}{\partial x^\alpha} = 0;$$

this equation admits $n-1$ independent solutions $f^2, \ldots, f^n$ which are analytic functions of the coordinates x^α of an n-dimensional portion $\mathscr{S}$ of $\mathscr{R}$. Hence the equations

$$\bar{x}^\alpha = f^\alpha(x),$$

where f^1 is an analytic function in $\mathscr{S}$ such that the determinant

$$\begin{vmatrix} \dfrac{\partial f^1}{\partial x^1} & & \dfrac{\partial f^n}{\partial x^1} \\ & & \\ & & \\ \dfrac{\partial f^1}{\partial x^n} & & \dfrac{\partial f^n}{\partial x^n} \end{vmatrix}$$

does not vanish in $\mathscr{S}$, define a transformation of the coordinates of $\mathscr{S}$ such that the components $\xi^\alpha(x)$ become $\bar{\xi}^1(x) \neq 0$ and $\bar{\xi}^\alpha = 0$ for $\alpha = 2, \ldots, n$. We now make a second coordinate transformation, namely

$$\bar{\bar{x}}^1 = \int \frac{d\bar{x}^1}{\bar{\xi}^1(\bar{x})}, \quad \bar{\bar{x}}^2 = \bar{x}^2, \ldots, \bar{\bar{x}}^n = \bar{x}^n,$$

where we suppose that the domain $\mathscr{S}$ is taken so small that the quantity $\bar{\xi}^1(\bar{x})$ is not equal to zero at any point of $\mathscr{S}$. This will then transform the components $\bar{\xi}^\alpha$ into the components $\bar{\bar{\xi}}^\alpha = \delta^\alpha_1$ with respect to the coordinates $\bar{\bar{x}}^\alpha$ of the domain $\mathscr{S}$.

11. Invariants

In the further development of the spatial theory of Chapter I we shall be concerned with the determination of tensors whose components depend on the *fundamental functions* and their derivatives for the space in question. For example, in the case of the affine space discussed in § 2 of Chapter I, we shall be concerned with tensors whose components depend on the fundamental functions $L^\alpha_{\beta\gamma}$ and their derivatives, i.e. more precisely on affine differential invariants in the sense of the following

Definition of Affine Tensor Differential Invariant. A tensor T will be called an affine differential invariant of order r if its components

$$T^{\alpha\ldots\beta}_{\gamma\ldots\delta}\left(L^\alpha_{\beta\gamma};\ \frac{\partial L^\alpha_{\beta\gamma}}{\partial x^\delta};\ \qquad \frac{\partial^r L^\alpha_{\beta\gamma}}{\partial x^\delta \ldots \partial x^\epsilon}\right)$$

are functions of the L's and their derivatives to the rth order, such that each component retains its form as a function of the L's and their derivatives under the transformation (10.1).

Replacing the law of transformation (10.1) in the above definition by other equations of transformation possessing the required property of transitivity, we are led to the concept of differential invariants which generalize the affine tensor differential invariants; we shall refer to such invariants as *non-tensor differential invariants*. Thus the quantities $\Omega^{\alpha}_{\beta\gamma}$ (2.4 (b)) are the components of a tensor differential invariant of order zero which is skew-symmetric in the indices β and γ; also the $L^{\alpha}_{\beta\gamma}$ are the components of a non-tensor differential invariant of order zero, i.e. *the affine connection*.

In the case of the metric space we shall be concerned with invariants in the general sense of the affine differential invariants except that the components of the invariants in question depend on the components $g_{\alpha\beta}$ and their derivatives; if the components of the invariant involve derivatives of the $g_{\alpha\beta}$ up to and including those of order r ($\geqq 0$), it will be called a *metric differential invariant of order r*. As an example of such invariants we have the *fundamental metric tensor* with components $g_{\alpha\beta}$. There are also the quantities $g^{\alpha\beta}$ defined by (5.4) which constitute the components of the contravariant form of the fundamental metric tensor. Likewise there are the Christoffel symbols (5.10) which constitute the components of a non-tensor differential invariant of order 1, i.e. the affine connection of this space.

When we consider the analogous differential invariants for a space of distant parallelism we must take account of transformations of the fundamental vectors of the type (6.4) as well as coordinate transformations; this leads to the following

Definition of Tensor Differential Invariant of a Space of Distant Parallelism. A tensor T will be called a tensor differential invariant of a space of distant parallelism of order r if its components

$$T^{\alpha\ldots\beta}_{\gamma\ldots\delta}\left(h^i_\alpha;\ \frac{\partial h^i_\alpha}{\partial x^\beta};\ \ldots;\ \frac{\partial^r h^i_\alpha}{\partial x^\beta\ldots\partial x^\gamma}\right)$$

are functions of the h^i_α and their derivatives to the rth order, such that (1) *each component retains its form as a function of the h^i_α and their derivatives under the transformation* (10.1) *and* (2) *each component remains unaltered by a transformation* (6.4) *of the fundamental vectors.*

The possibility of constructing a set of scalars as defined by (10.8) demands likewise that we define a set of scalar differential invariants of order r with components

$$U^{i\ldots j}_{k\ldots l}\left(h^i_\alpha;\ \frac{\partial h^i_\alpha}{\partial x^\beta};\ \ldots;\ \frac{\partial^r h^i_\alpha}{\partial x^\beta\ldots\partial x^\gamma}\right)$$

by the requirement that each component retains its form as a function of the h^i_α and their derivatives (1) when transformed as a scalar under a coordinate transformation and (2) when transformed by the transformation (10.9) imposed by the transformation (6.4) of the fundamental vectors.

Similarly we can consider differential invariants of a space of distant parallelism which are a combination of the two types above defined and for which the components depend both on Latin and Greek indices; such invariants, which are defined in an evident manner on the basis of the above definitions, will be called mixed differential invariants of a space of distant parallelism. The fundamental vectors themselves furnish an example of mixed invariants of order 0. As an example of a non-tensor differential invariant of order 1 we have the affine connection of the space of distant parallelism whose components are defined by (6.3).

We now consider the Weyl space. When the gauge is changed the covariant components of the fundamental metric tensor change in accordance with (8.2). Under this same change of gauge the equations

$$g^{*\alpha\beta} = \lambda^{-1}(x)\, g^{\alpha\beta} \tag{11.1}$$

represent the change in the contravariant components of the fundamental metric tensor. In the following definition we extend this idea to differential invariants of higher order.

Definition of Weyl Tensor Differential Invariant. A tensor T will be called a Weyl tensor differential invariant of order (r, s) and gauge-weight e if its components

$$T^{\alpha\ldots\beta}_{\gamma\ldots\delta}\Big(g \qquad \partial x^{\gamma}, \qquad \frac{\partial^r g_{\alpha\beta}}{\partial x^{\gamma}\ldots\partial x^{\delta}};\ \phi_k;\ \frac{\partial\phi_k}{\partial x^{\alpha}}; \quad \cdot\ \frac{\partial^s\phi_k}{\partial x^{\alpha}\ldots\partial x^{\beta}}\Big)$$

are functions of the $g_{\alpha\beta}$ and their derivatives to the rth order and of the ϕ_k and their derivatives to the sth order such that each component (1) *retains its form as a function of the above quantities under the transformation* (10.1) *and* (2) *is multiplied by λ^e as the result of a transformation of gauge, i.e. more precisely as the result of the transformations* (8.2) *and* (8.5).

Thus the covariant form of the fundamental metric tensor is of gauge-weight 1 and the contravariant form of this tensor is of gauge-weight -1. An example of a non-tensor differential invariant of order (1, 0) and gauge-weight 0 is furnished by the affine connection of the Weyl space.

If a contravariant index of the symbol of the components of a tensor T of gauge-weight e is lowered by means of the components $g_{\alpha\beta}$ as described in § 10, the resulting components will define a tensor T^* of gauge-weight $e+1$; similarly we can obtain a tensor T^* of gauge-weight $e-1$ by raising a covariant index of the symbol of the components of a tensor T of gauge-weight e. An interesting invariant offering the possibility of additional change of gauge-weight of a tensor T is the scalar density with component $\sqrt{g}$, where g is the determinant of the components $g_{\alpha\beta}$; this invariant has gauge-weight $\frac{n}{2}$. *By multiplying the components of a tensor T by the quantity $\sqrt{g}$ we obtain the components of a tensor T^* of gauge-weight $\frac{n}{2}$ greater than that of the tensor T; the weight of the tensor T^* so obtained will be $W+1$ if the weight of the original tensor T is W* (see § 10).

We shall see that the analytical theory of the affine space, based on the intrinsic differential invariants of this space, may be carried over to the metric space, the space of distant parallelism and the Weyl space without changing its essential characteristics. This circumstance is, in fact, indicated to a certain extent by the very terminology which we have adopted: for while we have used the designations of metric invariants, Weyl invariants and invariants of the space of distant parallelism in the statement of the above definitions, these tensors come under the general classification of affine tensors as defined in § 10.

12. Parallel displacement of a vector around an infinitesimal closed circuit

When we displace a vector by infinitesimal parallel displacement around an infinitesimal closed circuit in an affine space, the components of the vector will be altered in general. In deducing the analytical expression for the change in the components of the vector due to this displacement we arrive at an important differential invariant called the *affine curvature tensor* (3). If the components of this tensor vanish identically throughout the region $\mathscr{R}$, then the above parallel displacement of a vector around an infinitesimal closed circuit in this region will leave the components of the vector unaltered. In this sense the curvature tensor provides a measure of the curvature of space; a further discussion of the significance of this tensor is to be found in the later chapters.

We consider a point P of the region $\mathscr{R}$ through which passes an analytic surface of parametric representation

$$x^\alpha = x^\alpha(s,t), \tag{12.1}$$

the parameters s and t being chosen so that $s=0$, $t=0$ is the point P itself. Let ξ denote an arbitrary contravariant vector at P having components $(\xi^\alpha)_0$. Then the equations

$$\frac{d\xi^\alpha}{ds} + L^\alpha_{\beta\gamma}\xi^\beta \frac{dx^\gamma}{ds} = 0, \tag{12.2}$$

expressing the condition that the vectors with components ξ^α along the parametric line $t=0$ be parallel with respect to this curve, admit one and only one solution $\xi^\alpha(s,0)$ such that

$$\xi^\alpha(0,0) = (\xi^\alpha)_0. \tag{12.3}$$

Similarly the equations

$$\frac{d\xi^\alpha}{dt} + L^\alpha_{\beta\gamma}\xi^\beta \frac{dx^\gamma}{dt} = 0, \tag{12.4}$$

which express the condition that the vectors with components ξ^α along the parametric line $s=0$ be parallel, admit one and only one solution $\xi^\alpha(0,t)$ satisfying the above condition (12.3).

When we consider the functional components $\xi^\alpha(s, 0)$ and $\xi^\alpha(0, t)$ we see that there are two possibilities of extending the corresponding vectors by infinitesimal parallel displacements to the other points of the surface (12.1). For a given value of $t = t_0$ we may displace the vector with components $\xi^\alpha(0, t)$ in a parallel manner along the curve $t = t_0$. Allowing t_0 to assume all values in a small neighbourhood of $t = 0$, we obtain a vector ξ with components $\xi^\alpha(s, t)$ at each point of the surface (12.1). The similar process by which the vector having components $\xi^\alpha(s, 0)$ is displaced parallel to itself along the curve $s = s_0$ leads to a second set of vectors defined over the surface (12.1) with components which we shall denote by $\xi^\alpha_\star(s, t)$.

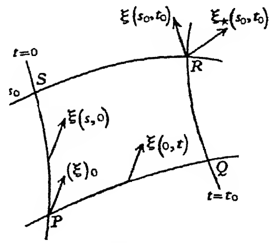

Fig. 6.

The difference of the components $\xi^\alpha_\star(s_0, t_0) - \xi^\alpha(s_0, t_0)$ represents the change in the components $\xi^\alpha_\star(s_0, t_0)$ due to infinitesimal parallel displacement around the circuit $RSPQR$ indicated in Fig. 6.

The method of derivation of the components $\xi^\alpha(s, t)$ and $\xi^\alpha_\star(s, t)$ shows that for

$$\begin{aligned} s = 0: &\quad \xi^\alpha(s, t) \to \xi^\alpha(0, t), \quad \xi^\alpha_\star(s, t) \to \xi^\alpha(0, t), \\ t = 0: &\quad \xi^\alpha(s, t) \to \xi^\alpha(s, 0), \quad \xi^\alpha_\star(s, t) \to \xi^\alpha(s, 0). \end{aligned}$$

In view of this fact*

$$\xi^\alpha_\star - \xi^\alpha = st\left\{\left(\frac{\partial^2(\xi^\alpha_\star - \xi^\alpha)}{\partial s\,\partial t}\right)_0 + \cdots\right\}, \tag{12.5}$$

where the derivatives in the right member are evaluated at the point $s = 0$, $t = 0$, and where the dots indicate terms multiplied by s, t. Hence

$$\lim_{s,t\to 0} \frac{(\xi^\alpha_\star - \xi^\alpha)}{st} = \left(\frac{\partial^2(\xi^\alpha_\star - \xi^\alpha)}{\partial s\,\partial t}\right)_0. \tag{12.6}$$

Since $\xi^\alpha_\star - \xi^\alpha$ are the components of a vector, it follows that the limit in (12.6) forms the components of a vector, and hence

$$\left(\frac{\partial^2(\xi^\alpha_\star - \xi^\alpha)}{\partial s\,\partial t}\right)_0$$

constitute the components of a vector.

* That the functions $\xi^\alpha_\star(s, t)$ are analytic in the variables s, t can be shown in the following manner. First observe that the functions $\xi^\alpha(s, 0)$ are analytic in the single variable s in consequence of their definition as solutions of the differential equations (12.2). Then the functions $\xi^\alpha_\star(s, t)$ are given as solutions of (12.8) by the power series expansions

$$\xi^\alpha_\star(s, t) = \xi^\alpha(s, 0) - \left[\left(L^\alpha_{\beta\gamma}\right)_{t=0} \xi^\beta(s, 0)\left(\frac{\partial x^\gamma}{\partial t}\right)_{t=0}\right] t - \cdots.$$

The coefficients of t in these expansions are analytic functions of the variable s on account of the analyticity of the components $L^\alpha_{\beta\gamma}$ and the surface (12.1); it is easily seen likewise that the coefficients of higher powers of t in these expansions are analytic functions of the variable s. Hence we can replace the first set of terms $\xi^\alpha(s, 0)$ and the successive coefficients in the above expansions by their corresponding power series. This results in the power series expansions of the functions $\xi^\alpha_\star(s, t)$ in the two variables s, t. Similar remarks apply to the functions $\xi^\alpha(s, t)$.

The components $\xi^\alpha(s,t)$ and $\xi^\alpha_\star(s,t)$ satisfy the equations

$$\frac{\partial \xi^\alpha}{\partial s} + L^\alpha_{\beta\gamma}\xi^\beta \frac{\partial x^\gamma}{\partial s} \equiv 0, \tag{12.7}$$

and

$$\frac{\partial \xi^\alpha_\star}{\partial t} + L^\alpha_{\beta\gamma}\xi^\beta_\star \frac{\partial x^\gamma}{\partial t} \equiv 0, \tag{12.8}$$

identically; likewise the equations

$$\frac{\partial \xi^\alpha}{\partial t} + L^\alpha_{\beta\gamma}\xi^\beta \frac{\partial x^\gamma}{\partial t} = 0 \quad (s=t=0), \tag{12.9}$$

and

$$\frac{\partial \xi^\alpha_\star}{\partial s} + L^\alpha_{\beta\gamma}\xi^\beta_\star \frac{\partial x^\gamma}{\partial s} = 0 \quad (s=t=0), \tag{12.10}$$

are satisfied at the point P. If we differentiate (12.7) with respect to t, evaluate at $s=t=0$, and then make use of (12.9), we obtain

$$\left(\frac{\partial^2 \xi^\alpha}{\partial s\,\partial t}\right)_0 = \left[-\frac{\partial L^\alpha_{\beta\gamma}}{\partial x^\delta}\xi^\beta \frac{\partial x^\gamma}{\partial s}\frac{\partial x^\delta}{\partial t} - L^\alpha_{\beta\gamma}\xi^\beta \frac{\partial^2 x^\gamma}{\partial s\,\partial t} + L^\alpha_{\mu\gamma}L^\mu_{\beta\delta}\xi^\beta \frac{\partial x^\gamma}{\partial s}\frac{\partial x^\delta}{\partial t}\right]_0. \tag{12.11}$$

Similarly

$$\left(\frac{\partial^2 \xi^\alpha_\star}{\partial s\,\partial t}\right)_0 = \left[-\frac{\partial L^\alpha_{\beta\gamma}}{\partial x^\delta}\xi^\beta \frac{\partial x^\gamma}{\partial t}\frac{\partial x^\delta}{\partial s} - L^\alpha_{\beta\gamma}\xi^\beta \frac{\partial^2 x^\gamma}{\partial s\,\partial t} + L^\alpha_{\mu\gamma}L^\mu_{\beta\delta}\xi^\beta \frac{\partial x^\gamma}{\partial t}\frac{\partial x^\delta}{\partial s}\right]_0, \tag{12.12}$$

by differentiating (12.8) with respect to s, evaluating at $s=t=0$, and then using (12.10). Hence

$$\left(\frac{\partial^2(\xi^\alpha_\star - \xi^\alpha)}{\partial s\,\partial t}\right)_0 = \left[B^\alpha_{\beta\gamma\delta}\xi^\beta \frac{\partial x^\gamma}{\partial s}\frac{\partial x}{\partial t}\right. \tag{12.13}$$

where

$$B^\alpha_{\beta\gamma\delta} = \frac{\partial L^\alpha_{\beta\gamma}}{\partial x^\delta} - \frac{\partial L^\alpha_{\beta\delta}}{\partial x^\gamma} + L^\alpha_{\mu\delta}L^\mu_{\beta\gamma} - L^\alpha_{\mu\gamma}L^\mu_{\beta\delta}. \tag{12.14}$$

The left members of (12.13), as well as the quantities $(\xi^\alpha)_0$, $(\partial x^\beta/\partial s)_0$ and $(\partial x^\gamma/\partial t)_0$ occurring in the right members of these equations, form the components of vectors. In consequence it can easily be shown, by recourse to the law of transformation of the components of the above vectors, that the quantities $B^\alpha_{\beta\gamma\delta}$ constitute the components of a tensor invariant, i.e. the affine curvature tensor above mentioned. It is seen immediately from (12.14) that the components $B^\alpha_{\beta\gamma\delta}$ are skew-symmetric in the indices γ and δ, i.e. the equations

$$B^\alpha_{\beta\gamma\delta} = -B^\alpha_{\beta\delta\gamma}$$

are satisfied identically; for other identities satisfied by the components of the curvature tensor derived on the basis of a general investigation of spatial identities, the reader is referred to Chapter VI.

From (12.6) and (12.13) we have

$$(12.15) \qquad \lim_{s,t\to 0} \frac{\qquad \xi^{\alpha})}{st} = \left[B^{\alpha}_{\beta\gamma\delta} \xi^{\beta} \frac{\partial x^{\gamma}}{\partial s} \frac{\partial x^{\delta}}{\partial t} \right]_0 .$$

Now put

$$(12.16) \qquad \Delta\xi^{\alpha} = \xi^{\alpha}_{\star} - \xi^{\alpha}.$$

Also denote the differential components of an infinitesimal displacement from the point P along the curve $t=0$ by dx^{α} and along the curve $s=0$ by δx^{α}; let us in fact identify these displacements with the displacements PS and PQ respectively (see Fig. 6). *Then*

$$(12.17) \qquad \Delta\xi^{\alpha} = B^{\alpha}_{\beta\gamma\delta} \xi^{\beta} dx^{\gamma} \delta x^{\delta},$$

where the quantities $\Delta\xi^{\alpha}$ give the change in the components $(\xi^{\alpha})_0$ when the corresponding vector is carried by infinitesimal parallel displacement around the closed circuit $(dx,\ \delta x)$, i.e. the circuit $PQRSP$, the components $B^{\alpha}_{\beta\gamma\delta}$ and ξ^{α} being evaluated at the point P, and where terms of higher order in the right members of (12.17) *than those written down explicitly are neglected.* The direct interpretation of equations (12.17) on the basis of the preceding work is that the quantities $\Delta\xi^{\alpha}$ represent the change in the components $\xi^{\alpha}_{\star}(s_0, t_0)$ of the vector at R when carried around the circuit $RSPQR$ by infinitesimal parallel displacement, the components $B^{\alpha}_{\beta\gamma\delta}$ and ξ^{α} being evaluated at the point P. We can, however, consider the components $B^{\alpha}_{\beta\gamma\delta}$ and ξ^{α} in (12.17) to be evaluated at the point R, since this will cause the terms in the right members of (12.17) to change by infinitesimals of higher order than these terms themselves; then there is no longer a distinction between the points P and R and we are justified in making the above italicized statement.

A similar displacement of a covariant vector around the infinitesimal circuit $(dx,\ \delta x)$ leads to the equations

$$(12.18) \qquad \Delta\mu_{\beta} = -B^{\alpha}_{\beta\gamma\delta} \mu_{\alpha} dx^{\gamma} \delta x^{\delta}$$

in place of (12.17). While these latter equations can be deduced in a manner analogous to that by which (12.17) was obtained, we can also obtain them directly from (12.17) in the following manner. Since the quantity $\mu_{\sigma}\xi^{\sigma}$ is unaltered under infinitesimal parallel displacement (see § 2), we have that

$$(12.19) \qquad \xi^{\sigma}\Delta\mu_{\sigma} + \mu_{\sigma}\Delta\xi^{\sigma} = 0,$$

where the quantities in this equation have the same significance as in (12.18) and (12.17). Since the components ξ^{σ} in (12.19) can have arbitrary values, we can put $\xi^{\sigma} = \delta^{\sigma}_{\alpha}$; making this substitution and also eliminating $\Delta\xi^{\sigma}$ from (12.19) by means of (12.17), the equations (12.18) result.

A direct proof of the tensor character of the quantities $B^{\alpha}_{\beta\gamma\delta}$ defined by (12.14) can be made in the following manner. Write equations (9.4) in the form

$$\bar{L}^{\lambda}_{\mu\nu} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\lambda}} = \frac{\partial^2 x^{\alpha}}{\partial \bar{x}^{\mu} \partial \bar{x}^{\nu}} + L^{\alpha}_{\beta\gamma} \frac{\partial x^{\beta}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\nu}}.$$

Differentiate these equations with respect to $\bar{x}^\xi$, interchange the indices ν and ξ and subtract thereby eliminating third derivatives; from the equations so obtained, use the above equations to eliminate second derivatives. Then we have a system of equations which can be written

$$\bar{B}^\lambda_{\mu\nu\xi}\frac{\partial x^\alpha}{\partial \bar{x}^\lambda} = B^\alpha_{\beta\gamma\delta}\frac{\partial x^\beta}{\partial \bar{x}^\mu}\frac{\partial x^\gamma}{\partial \bar{x}^\nu}\frac{\partial x^\delta}{\partial \bar{x}^\xi},$$

where the quantities $B^\alpha_{\beta\gamma\delta}$ are defined by (12.14).

The above discussion also applies to the cases of the metric space, the space of distant parallelism and the Weyl space, in each of which we have an analogous definition of infinitesimal parallel displacement. In the case of the metric and Weyl spaces the affine curvature tensor admits an associated tensor with components

$$B_{\alpha\beta\gamma\delta} = g_{\alpha\sigma}B^\sigma_{\beta\gamma\delta}; \tag{12.20}$$

we shall find that it is sometimes of advantage to consider this associated tensor in our later work. It is of particular interest to notice in this connection that the affine curvature tensor for the space of distant parallelism vanishes identically, i.e.

$$\frac{\partial \Delta^\alpha_{\beta\gamma}}{\partial x^\delta} - \frac{\partial \Delta^\alpha_{\beta\delta}}{\partial x^\gamma} + \Delta^\alpha_{\mu\delta}\Delta^\mu_{\beta\gamma} - \Delta^\alpha_{\mu\gamma}\Delta^\mu_{\beta\delta} = 0,$$

as can readily be verified by substitution into these equations of the components $\Delta^\alpha_{\beta\gamma}$ defined by (6.3). As a matter of fact the vanishing of the components of this curvature tensor is a direct consequence of a result stated at the end of § 6 in Chapter I and so can be inferred without additional calculation.

We see that the affine curvature tensor with components $B^\alpha_{\beta\gamma\delta}$ is a differential invariant of orders 1 and 2 in the affine and metric spaces, respectively; for the Weyl space this invariant is of order (2.1) and of gauge-weight 0.

13. Covariant differentiation

Covariant differentiation is a process, applicable in a space bearing an affine connection, by which we can deduce from a given tensor T a new tensor $T^\star$ involving an additional covariant index in the symbol of its components; the tensor $T^\star$ is called the *covariant derivative* of the tensor T from analogy between the processes of covariant differentiation and partial differentiation(4). Let us first observe that

$$(13.1)\qquad \begin{aligned} &(a)\quad \frac{\partial(x\bar{x})}{\partial \bar{x}^\alpha} = (x\bar{x})\frac{\partial^2 x^\gamma}{\partial \bar{x}^\alpha \partial \bar{x}^\beta}\frac{\partial \bar{x}^\beta}{\partial x^\gamma},\\ &(b)\quad \bar{L}^\alpha_{\beta\alpha} = L^\alpha_{\gamma\alpha}\frac{\partial x^\gamma}{\partial \bar{x}^\beta} + \frac{\partial \log(x\bar{x})}{\partial \bar{x}^\beta} \end{aligned}$$

Then differentiating (10.1) with respect to x^ω, making use of (13.1), and eliminating the second derivatives which arise by means of equations of the type (9.4), namely

$$\bar{L}^{\lambda}_{\mu\nu}\frac{\partial x^{\alpha}}{\partial \bar{x}^{\lambda}}=\frac{\partial^2 x^{\alpha}}{\partial \bar{x}^{\mu}\,\partial \bar{x}^{\nu}}+L^{\alpha}_{\beta\gamma}\frac{\partial x^{\beta}}{\partial \bar{x}^{\mu}}\frac{\partial x^{\gamma}}{\partial \bar{x}^{\nu}}, \tag{13.2}$$

$$L^{\lambda}_{\mu\nu}\frac{\partial \bar{x}^{\alpha}}{\partial x^{\lambda}}=\frac{\partial^2 \bar{x}^{\alpha}}{\partial x^{\mu}\,\partial x^{\nu}}+\bar{L}^{\alpha}_{\beta\gamma}\frac{\partial \bar{x}^{\beta}}{\partial x^{\mu}}\frac{\partial \bar{x}^{\gamma}}{\partial x^{\nu}}, \tag{13.3}$$

we obtain

$$\bar{T}^{\lambda\dots\mu}_{\nu\dots\xi,\omega}=(x\bar{x})^W\,T^{\alpha\dots\beta}_{\gamma\dots\delta,\epsilon}\frac{\partial \bar{x}^{\lambda}}{\partial x^{\alpha}}\cdots\frac{\partial \bar{x}^{\mu}}{\partial x^{\beta}}\frac{\partial x^{\gamma}}{\partial \bar{x}^{\nu}}\cdot\,\cdot\frac{\partial x^{\epsilon}}{\partial \bar{x}^{\omega}}, \tag{13.4}$$

where

$$\begin{aligned}T^{\alpha\dots\beta}_{\gamma\dots\delta,\epsilon}=\frac{\partial T^{\alpha\dots\beta}_{\gamma\dots\delta}}{\partial x^{\epsilon}}-T^{\alpha\dots\beta}_{\sigma\dots\delta}L^{\sigma}_{\gamma\epsilon}-\dots-T^{\alpha\dots\beta}_{\gamma\dots\sigma}L\\ +T^{\sigma\dots\beta}_{\gamma\dots\delta}L^{\alpha}_{\sigma\epsilon}+\dots+T^{\alpha\dots\sigma}_{\gamma\dots\delta}L^{\beta}_{\sigma\epsilon}-WT^{\alpha\dots\beta}_{\gamma\dots\delta}L^{\sigma}_{\sigma\epsilon},\end{aligned} \tag{13.5}$$

and there is an analogous expression for the quantities in the left members of (13.4); hereafter it will be understood without special mention that when an expression of the type (13.5) is written down, an analogous expression exists in the barred quantities. By (13.4) we see that the tensor $T^{\star}$ with components $T^{\alpha\dots\beta}_{\gamma\dots\delta,\epsilon}(x)$, i.e. the covariant derivative of the tensor T, is a relative affine tensor of weight W.*

In particular if T is a relative scalar of weight W so that the transformation equations

$$\bar{T}=(x\bar{x})^W\,T$$

apply, the above process gives

$$\bar{T}_{,\omega}=(x\bar{x})^W\,T_{,\epsilon}\frac{\partial x^{\epsilon}}{\partial \bar{x}^{\omega}},$$

where

$$T_{,\omega}=\frac{\partial T}{\partial x^{\omega}}-WTL^{\alpha}_{\alpha\omega}; \tag{13.6}$$

* Equations (13.2) and (13.3) will remain valid if we interchange the lower indices on the components $L^{\alpha}_{\beta\gamma}$; hence (13.4) will continue to hold if the lower indices on the components $L^{\alpha}_{\beta\gamma}$ are interchanged in (13.5). It is not essential, however, to introduce the modified covariant derivative, which thereby arises, into our theory in addition to the above covariant derivative $T^{\star}$.

We may observe also that the quantities $L^{\alpha}_{\alpha\omega}$ in the last term of the expression defining the components (13.5) can be replaced by $L^{\alpha}_{\omega\alpha}$ without changing the tensor character of these components. To indicate the components of this modification of the above covariant derivative we may, for example, replace the comma (,) in (13.5) by a bar (/).

A generalization of the above process of covariant differentiation was suggested by W. Mayer in the treatise by Duschek-Mayer, *Lehrbuch der Differentialgeometrie* (B. G. Teubner, 1930), **2**, Chapter VII; this method was developed and applied to the treatment of subspaces by J. A. Schouten and E. R. van Kampen, "Zur Einbettungs- und Krummungstheorie nichtholonomer Gebilde", *Math. Ann.* **103** (1930), pp. 752–83. See also A. W. Tucker, "On generalized covariant differentiation", *Ann. of Math.* (2), **32** (1931), pp. 451–60; E. H. Cutler, "Frenet formulas for a general subspace of a Riemann space", *Trans. Amer. Math. Soc.* **33** (1931), pp. 839–50.

equations (13.5) are therefore to be regarded as reducing to these latter equations (13.6) when T is a relative scalar of weight W.

Repeating the process of covariant differentiation we obtain the successive covariant derivatives of the tensor T; we call these the second, third, ... covariant derivatives of T. Thus

$$T^{\alpha\dots\beta}_{\gamma\dots\delta,\epsilon};\quad T^{\alpha\dots\beta}_{\gamma\dots\delta,\epsilon,\zeta};\quad T^{\alpha\dots\beta}_{\gamma\dots\delta,\epsilon,\zeta,\eta};\quad \dots$$

are the components of the first, second, third, ... covariant derivatives of the tensor T.

It is easy to see that covariant differentiation of the sums, differences, or products of tensors obeys rules analogous to those of partial differentiation. For example

$$C^{\alpha\dots\beta}_{\gamma\dots\delta,\epsilon}=A^{\alpha\dots\beta}_{\gamma\dots\delta,\epsilon}+B^{\alpha\dots\beta}_{\gamma\dots\delta,\epsilon},$$
$$C^{\alpha\dots\beta\,\lambda\dots\mu}_{\gamma\dots\delta\,\nu\dots\xi,\rho}=A^{\alpha\dots\beta}_{\gamma\dots\delta,\rho}B^{\lambda\dots\mu}_{\nu\dots\xi}+A^{\alpha\dots\beta}_{\gamma\dots\delta}B^{\lambda\dots\mu}_{\nu\dots\xi,\rho},$$
$$B^{\beta\dots\gamma}_{\mu\dots\nu,\xi}=A^{\alpha\beta\dots\gamma}_{\alpha\mu\dots\nu,\xi},$$

from (10.2), (10.3) and (10.4), respectively.

1°. A generalization of the *Ricci identities* of Riemann geometry can be deduced from (13.5). Let us observe first that if Ψ is the component of a relative scalar of weight W, the components of its second covariant derivative are

$$\Psi_{,\alpha,\beta}=\frac{\partial\Psi_{,\alpha}}{\partial x^{\beta}}-\Psi_{,\epsilon}L^{\epsilon}_{\alpha\beta}-W\Psi_{,\alpha}L^{\epsilon}_{\epsilon\beta};$$

hence

$$\Psi_{,\alpha,\beta}-\Psi_{,\beta,\alpha}=-2\frac{\partial\Psi}{\partial x^{\sigma}}\Omega^{\sigma}_{\alpha\beta}-W\Psi B^{\gamma}_{\gamma\alpha\beta}.$$

Likewise

(13.7) $$\xi^{\alpha}_{,\beta,\gamma}-\xi^{\alpha}_{,\gamma,\beta}=\xi^{\sigma}B^{\alpha}_{\sigma\beta\gamma}-2\xi^{\alpha}_{,\sigma}\Omega^{\sigma}_{\beta\gamma}-W\xi^{\alpha}B^{\epsilon}_{\epsilon\beta\gamma},$$

(13.8) $$\lambda_{\alpha,\beta,\gamma}-\lambda_{\alpha,\gamma,\beta}=-\lambda_{\sigma}B^{\sigma}_{\alpha\beta\gamma}-2\lambda_{\alpha,\sigma}\Omega^{\sigma}_{\beta\gamma}-W\lambda_{\alpha}B^{\epsilon}_{\epsilon\beta\gamma},$$

(13.9) $$a_{\alpha\beta,\gamma,\delta}-a_{\alpha\beta,\delta,\gamma}=-a_{\mu\beta}B^{\mu}_{\alpha\gamma\delta}-a_{\alpha\mu}B^{\mu}_{\beta\gamma\delta}-2a_{\alpha\beta,\mu}\Omega^{\mu}_{\gamma\delta}-Wa_{\alpha\beta}B^{\epsilon}_{\epsilon\gamma\delta},$$

etc. In general if T represents any relative tensor of weight W, we have

(13.10) $$\begin{aligned}T^{\alpha\dots\beta}_{\gamma\dots\delta,\sigma,\tau}-T^{\alpha\dots\beta}_{\gamma\dots\delta,\tau,\sigma}&=-T^{\alpha\dots\beta}_{\mu\dots\delta}B^{\mu}_{\gamma\sigma\tau}-\dots-T^{\alpha\dots\beta}_{\gamma\dots\mu}B^{\mu}_{\delta\sigma\tau}\\&+T^{\mu\dots\beta}_{\gamma\dots\delta}B^{\alpha}_{\mu\sigma\tau}+\dots+T^{\alpha\dots\mu}_{\gamma\dots\delta}B^{\beta}_{\mu\sigma\tau}-2T^{\alpha\dots\beta}_{\gamma\dots\delta,\mu}\Omega^{\mu}_{\sigma\tau}-WT^{\alpha\dots\beta}_{\gamma\dots\delta}B^{\epsilon}_{\epsilon\sigma\tau}.\end{aligned}$$

These equations constitute the generalization of the above mentioned Ricci identities (5).

2°. There is an interesting formula which can be derived from (13.5) for the case where the tensor T is a contravariant vector density; denoting the components of this vector density by $\mathfrak{V}^{\alpha}$, we have

$$\mathfrak{V}^{\alpha}_{|\epsilon}=\frac{\partial\mathfrak{V}^{\alpha}}{\partial x^{\epsilon}}+\mathfrak{V}^{\sigma}L^{\alpha}_{\sigma\epsilon}-\mathfrak{V}^{\alpha}L^{\sigma}_{\epsilon\sigma}$$

(see above footnote). Hence

(13.11) $$\mathfrak{V}^{\alpha}_{|\alpha}=\frac{\partial\mathfrak{V}^{\alpha}}{\partial x^{\alpha}},$$

i.e. the contracted form of all the components of the covariant derivative of the contravariant vector density is merely the sum appearing in the right member of the above equation.

The tensor having components

$$T^{\alpha\cdots\beta}_{\gamma\cdots\delta,\beta} \qquad T^{\alpha\cdots\beta}_{\gamma\cdots\delta/\beta}$$

is called the *divergence of the tensor* T. Thus the quantity $\mathfrak{V}^{\alpha}_{/\alpha}$ in (13.11) is the component of the divergence of a vector density.

Equations (13.5) likewise furnish immediately certain simple identities related to metric space and the space of distant parallelism which are of particular interest. For example,

$$g_{\alpha\beta,\gamma}=\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}-g_{\sigma\beta}\Gamma^{\sigma}_{\alpha\gamma}-g_{\alpha\sigma}\Gamma^{\sigma}_{\beta\gamma}=0, \tag{13.12}$$

$$g^{\alpha\beta}_{,\gamma}=\frac{\partial g^{\alpha\beta}}{\partial x^{\gamma}}+g^{\alpha\sigma}\Gamma^{\beta}_{\sigma\gamma}+g^{\sigma\beta}\Gamma^{\alpha}_{\sigma\gamma}=0, \tag{13.13}$$

where the Γ's in these equations are Christoffel symbols. Since the determinant g of the components $g_{\alpha\beta}$ is itself the component of a relative scalar of weight 2 by (10.7), we can apply (13.6) to obtain the components of the covariant derivative of this scalar; this gives

$$g_{,\alpha}=\frac{\partial g}{\partial x^{\alpha}}-2g\Gamma^{\sigma}_{\alpha\sigma}=0, \tag{13.14}$$

identically. Hence using (5.4), (5.10) and (13.14), we can write

$$\Gamma^{\sigma}_{\alpha\sigma}=\tfrac{1}{2}\frac{\partial \log g}{\partial x^{\alpha}}=\tfrac{1}{2}g^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}=-\tfrac{1}{2}g_{\mu\nu}\frac{\partial g^{\mu\nu}}{\partial x^{\alpha}}. \tag{13.15}$$

Similarly we have the equations

$$h^{\alpha}_{i,\beta}=\frac{\partial h^{\alpha}_{i}}{\partial x^{\beta}}+h^{\sigma}_{i}\Delta^{\alpha}_{\sigma\beta}=0, \tag{13.16}$$

$$h^{i}_{\alpha,\beta}=\frac{\partial h^{i}_{\alpha}}{\partial x^{\beta}}-h^{i}_{\sigma}\Delta^{\sigma}_{\alpha\beta}=0, \tag{13.17}$$

defining the components of the covariant derivatives of the fundamental vectors of a space of distant parallelism; the components of these covariant derivatives vanish identically, the equations (13.16) and (13.17) being completely analogous to (13.12) and (13.13).

The process of covariant differentiation may be applied to the affine curvature tensor to obtain a sequence of tensor invariants with components

$$B^{\alpha}_{\beta\gamma\delta,\epsilon};\ B^{\alpha}_{\beta\gamma\delta,\epsilon,\zeta};\ \cdots \tag{13.18}$$

which are of fundamental importance in the problem of the characterization of spaces by means of their differential invariants (see Chapter VIII). In the case of the metric space the sequence (13.18) can be replaced by

$$B_{\alpha\beta\gamma\delta,\epsilon};\ B_{\alpha\beta\gamma\delta,\epsilon,\zeta};\ \cdots. \tag{13.19}$$

This latter sequence is deducible from the completely covariant form of the affine curvature tensor whose components are defined by (12.20).

As an example of a problem in which the above sequence of components (13.18) occurs, let us consider the problem of determining necessary and sufficient conditions for the existence of a field of parallel contravariant vectors in a general affinely connected space. Denoting the components of these vectors by ξ^α, we have the equations

$$\frac{\partial \xi^\alpha}{\partial x^\beta} + \xi^\mu L^\alpha_{\mu\beta} = 0 \tag{13.20}$$

as the required conditions, since these equations express the condition that the vectors of the field at points P and Q be parallel independently of the route of displacement C by which P and Q are joined. Differentiating (13.20) with respect to x^γ, interchanging β and γ and subtracting, we obtain the integrability conditions of these equations, namely

$$\xi^\mu B^\alpha_{\mu\beta\gamma} = 0. \tag{13.21a}$$

Successive covariant differentiation of (13.21a) leads to the following sequence of equations

$$\begin{cases} \xi^\mu B^\alpha_{\mu\beta\gamma,\,\delta} = 0, \\ \xi^\mu B^\alpha_{\mu\beta\gamma,\,\delta,\,\epsilon} = 0, \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \end{cases} \tag{13.21b}$$

in view of the fact that the quantities $\xi^\mu_{,\nu}$ vanish identically on account of (13.20).

A necessary condition for the existence of the above field of parallel vectors is that equations (13.21) be algebraically consistent when considered as equations for the determination of the components $\xi^\alpha(x)$; in particular it is evident that the first $p\ (\geqq 1)$ sets of equations (13.21) must be algebraically consistent and that all their solutions ξ^α must satisfy the $p+1$st set of these equations. Thus if

$$\xi^\alpha_{(i)}, \quad \begin{bmatrix} \alpha = 1, \dots, n \\ i = 1, \dots, q \\ 1 \leqq q \leqq n \end{bmatrix}$$

represents a fundamental set of linearly independent solutions of the first p sets of equations (13.21), the general solution of these equations can be written

$$\xi^\alpha = \phi^i \xi^\alpha_{(i)}, \tag{13.22}$$

where the expression on the right is summed on i for $i = 1, \dots, q$ and the ϕ's are arbitrary functions of the x coordinates; by hypothesis the functions ξ^α defined by (13.22) satisfy the $p+1$st set of equations (13.21). Hence if we substitute any one of the solutions $\xi^\alpha_{(i)}$ into the first p sets of equations (13.21), differentiate these equations covariantly, and then make use of the fact that $\xi^\alpha_{(i)}$ satisfies the $p+1$st set of these equations, we deduce that $\xi^\alpha_{(i),\,\sigma}$ likewise satisfies the first p sets of equations (13.21). Hence we can write

$$\xi^\alpha_{(i),\,\alpha} = \lambda^k_{i\sigma}\, \xi^\sigma_{(k)} \tag{13.23}$$

for suitably chosen functions $\lambda^k_{i\sigma}$, where $i, k = 1, \dots, q$; $\sigma = 1, \dots, n$. By (13.7) and (13.21a) we have

$$\xi^\alpha_{(i),\,\sigma,\,\tau} - \xi^\alpha_{(i),\,\tau,\,\sigma} = -\xi^\beta_{(i)} B^\alpha_{\beta\sigma\tau} - 2\xi^\alpha_{(i),\,\beta}\, \Omega^\beta_{\sigma\tau} = -2\xi^\alpha_{(i),\,\beta}\, \Omega^\beta_{\sigma\tau}. \tag{13.24}$$

On account of (13.23) we deduce from these latter equations that

$$\left[\frac{\partial \lambda^k_{i\sigma}}{\partial x^\tau} - \frac{\partial \lambda^k_{i\tau}}{\partial x^\sigma} + \lambda^l_{i\sigma}\lambda^k_{l\tau} - \lambda^l_{i\tau}\lambda^k_{l\sigma}\right] \xi^\alpha_{(k)} = 0,$$

where $i, k, l = 1, \dots, q$; hence

$$\frac{\partial \lambda^k_{i\sigma}}{\partial x^\tau} - \frac{\partial \lambda^k_{i\tau}}{\partial x^\sigma} + \lambda^l_{i\sigma}\lambda^k_{l\tau} - \lambda^l_{i\tau}\lambda^k_{l\sigma} = 0, \tag{13.25}$$

$$\begin{bmatrix} i, k, l = 1, \dots, q \\ \sigma, \tau = 1, \dots, n \end{bmatrix}$$

since the rank of the matrix $\xi^{\alpha}_{(k)}$ is q. Now we wish to choose the functions ϕ^i in (13.22) so that the quantities ξ^{α} defined by these equations will satisfy (13.20). On substituting the ξ^{α} given by (13.22) into (13.20) and making use of (13.23) we find that the required condition on the ϕ's is that

$$\frac{\partial \phi^k}{\partial x^{\beta}} + \phi^i \lambda^k_{i\beta} = 0. \tag{13.26}$$

But these latter equations are completely integrable, the conditions of integrability being satisfied in consequence of (13.25); hence (13.26) possesses a unique solution $\phi^i(x)$ determined when the arbitrary initial values of the q functions ϕ^i are assigned. We can therefore state the following(6)

THEOREM. *A necessary and sufficient condition for the existence of a field of parallel contravariant vectors in a general affinely connected space is that we can find an integer* p ($\geqq 1$) *such that the first* p *sets of equations* (13.21) *admit a fundamental set of* q *solutions* ($1 \leqq q \leqq n$) *which satisfy the* $p+1$*st set of these equations.*

It is evident from what we have said above regarding the solution of the equations (13.26) that if the conditions of this theorem are satisfied, there will exist q linearly independent fields of parallel vectors and any linear combination, with constant coefficients, of these vectors will constitute the components of a field of parallel vectors. We observe in particular that if $q=n$ the curvature tensor and hence its successive derivatives must vanish identically; in other words all coefficients B in the equations (13.21) will vanish. An analogous theorem can of course be stated for the existence of one or more fields of parallel covariant vectors.

14. ALTERNATIVE METHODS OF COVARIANT DIFFERENTIATION. EXTENSION

The above sequence of covariant derivatives whose components are given by (13.19) fails to exist in the case of the space of distant parallelism owing to the identical vanishing of the affine curvature tensor for this space; moreover the covariant derivatives of the fundamental vectors vanish identically on account of (13.16) and (13.17) so that here again we lack a starting point for the construction of a sequence analogous to (13.19). We are accordingly led to consider the possibility of a modification of the above process of covariant differentiation which will permit us to construct a sequence of fundamental invariants for the space of distant parallelism, the need for such a sequence of differential invariants arising in the characterization problem in Chapter VIII.

Now the above theory of covariant differentiation was based on the concept of infinitesimal parallel displacement. It is evident however that the concept of path as considered in Chapter I will likewise serve as a suitable basis for a theory of covariant differentiation. For a general affinely connected space the paths are defined by

$$\frac{d^2x^{\alpha}}{ds^2} + \Gamma^{\alpha}_{\beta\gamma}\frac{dx^{\beta}}{ds}\frac{dx^{\gamma}}{ds} = 0, \tag{14.1}$$

where the Γ's are the symmetric part of components of affine connection; these equations likewise define the paths in the case of the geometry of paths described in § 3, for then the quantities $\Gamma^{\alpha}_{\beta\gamma}$ are the components of symmetric

affine connection of the space. Moreover the above equations define the paths in a metric or Weyl space according as the components $\Gamma^{\alpha}_{\beta\gamma}$ are Christoffel symbols, or the quantities given by (8.11), respectively. For a space of distant parallelism however the paths are defined by the equations

$$\frac{d^2x^{\alpha}}{ds^2}+\Lambda^{\alpha}_{\beta\gamma}\frac{dx^{\beta}}{ds}\frac{dx^{\gamma}}{ds}=0, \tag{14.2}$$

in which the quantities Λ are given in terms of the components of affine connection $\Delta^{\alpha}_{\beta\gamma}$ by the equations

$$\Lambda^{\alpha}_{\beta\gamma}=\tfrac{1}{2}(\Delta^{\alpha}_{\beta\gamma}+\Delta^{\alpha}_{\gamma\beta})=\tfrac{1}{2}h^{\alpha}_{i}\left(\frac{\partial h^{i}_{\beta}}{\partial x^{\gamma}}+\frac{\partial h^{i}_{\gamma}}{\partial x^{\beta}}\right).$$

Now the quantities $\Gamma^{\alpha}_{\beta\gamma}$ and $\Lambda^{\alpha}_{\beta\gamma}$ occurring in (14.1) and (14.2), respectively, transform by equations of the type (13.2), namely

$$\bar{\Gamma}^{\lambda}_{\mu\nu}\frac{\partial x^{\alpha}}{\partial \bar{x}^{\lambda}}=\frac{\partial^2 x^{\alpha}}{\partial \bar{x}^{\mu}\partial \bar{x}^{\nu}}+\Gamma^{\alpha}_{\beta\gamma}\frac{\partial x^{\beta}}{\partial \bar{x}^{\mu}}\frac{\partial x^{\gamma}}{\partial \bar{x}^{\nu}}, \tag{14.3}$$

$$\bar{\Lambda}^{\lambda}_{\mu\nu}\frac{\partial x^{\alpha}}{\partial \bar{x}^{\lambda}}=\frac{\partial^2 x^{\alpha}}{\partial \bar{x}^{\mu}\partial \bar{x}^{\nu}}+\Lambda^{\alpha}_{\beta\gamma}\frac{\partial x^{\beta}}{\partial \bar{x}^{\mu}}\frac{\partial x^{\gamma}}{\partial \bar{x}^{\nu}}, \tag{14.4}$$

under transformations of coordinates. Hence we can replace the equations (13.2) by equations (14.3) or (14.4) and the equations (13.1 (b)) and (13.3) derived from (13.2) by the analogous equations derived from (14.3) or (14.4) in the process of determining the covariant derivative T^* described in § 13. The result will be a tensor T^{**} with components defined by equations of the type (13.5) in which the components L are replaced by the above components Γ or Λ respectively; this tensor T^{**} will be called the *first extension* of the original tensor T.* In the case of metric and Weyl spaces there is no distinction between the process of covariant differentiation and the process of first extension owing to the fact that the affine connection is symmetric. On the other hand in the general affinely connected space and in the space of distant parallelism the processes of covariant differentiation and first extension will in general be distinct.

Let us now consider the first extension of the fundamental vectors of the space of distant parallelism. We have

$$h^{i}_{\alpha,\beta}=\frac{\partial h^{i}_{\alpha}}{\partial x^{\beta}}-h^{i}_{\gamma}\Lambda^{\gamma}_{\alpha\beta}=\frac{1}{2}\left(\frac{\partial h^{i}_{\alpha}}{\partial x^{\beta}}-\frac{\partial h^{i}_{\beta}}{\partial x^{\alpha}}\right) \tag{14.5}$$

as the components of the extensions of the covariant forms of these vectors; these extensions have components given by

$$h^{i}_{j,k}=h^{i}_{\alpha,\beta}h^{\alpha}_{j}h^{\beta}_{k}, \tag{14.6}$$

* Higher extensions are defined in § 32.

with respect to the fundamental vector configurations of this space. Let us put

$$h^i_{j,k,l} = \frac{\partial h^i_{j,k}}{\partial x^\alpha} h^\alpha_l; \tag{14.7}$$

the quantities defined by these equations are the components with respect to the fundamental vector configurations of the first extensions of the scalar invariants whose components are given by (14.6). Continuing in this way we have the sequence of scalar differential invariants with components

$$h^i_{j,k};\ h^i_{j,k,l};\ h^i_{j,k,l,m};\ \ldots.$$

The first set of these scalar invariants, namely those invariants possessing the quantities $h^i_{j,k}$ as components, will be seen later to correspond in certain respects to the affine curvature tensor of a generally affinely connected space.

Whether the concept of infinitesimal parallel displacement or the concept of path is taken as basic in the process of deriving new tensors, e.g. covariant derivatives or extensions, it is clear that from a formal analytical standpoint the possibility of this process depends on the existence of a set of transformation equations of the type of (13.2); the existence of other sets of equations of this type will furnish the possibility of deriving new tensors from a given tensor T analogous to covariant derivatives and extensions.

In a space of distant parallelism we might, for example, consider a theory analogous to covariant differentiation based on the transformation equations (14.3) in which the Γ's are Christoffel symbols. This would give the more complicated quantities

$$\frac{\partial h^i_\alpha}{\partial x^\beta} - \tfrac{1}{2} h^i_\sigma g^{\sigma\gamma}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right)$$

in place of the components $h^i_{\alpha,\beta}$ defined by (14.5); there is, however, no necessity to introduce such a method of formation of tensors in the further development of the theory of the space of distant parallelism (see § 83).

The formation of other equations of the type of (13.2), (14.3) and (14.4) can easily be effected. For example, in the space of distant parallelism we can deduce

$$\frac{\bar{u}\,\bar{\Gamma}^\lambda_{\mu\nu} + \bar{v}\,\bar{\Lambda}^\lambda_{\mu\nu}}{\bar{u}+\bar{v}} \frac{\partial x^\alpha}{\partial \bar{x}^\lambda} = \frac{\partial^2 x^\alpha}{\partial \bar{x}^\mu\, \partial \bar{x}^\nu} + \frac{u\,\Gamma^\alpha_{\beta\gamma} + v\,\Lambda^\alpha_{\beta\gamma}}{u+v} \frac{\partial x^\beta}{\partial \bar{x}^\mu} \frac{\partial x^\gamma}{\partial \bar{x}^\nu},$$

in which u and v are arbitrary analytic functions of the coordinates x^α of the region $\mathfrak{R}$ and $u \to \bar{u}$, $v \to \bar{v}$ by the scalar transformation; in particular u and v can be taken as arbitrary constants. The above set of transformation equations will lead to derived tensors analogous to covariant derivatives; these equations likewise lead to the definition of a system of invariantive curves determined by the differential equations

$$\frac{d^2 x^\alpha}{ds^2} + \left[\frac{u\,\Gamma^\alpha_{\beta\gamma} + v\,\Lambda^\alpha_{\beta\gamma}}{u+v}\right] \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0.$$

15. Differential parameters

The notion of a tensor differential invariant defined in § 11 can be extended to those cases in which a set of arbitrary scalar functions $F^{(k)}(x^1, \ldots, x^n)$, where $k = 1, \ldots, w$, and their derivatives occur in the expressions for the components of the entity.

Definition of Affine Tensor Differential Parameter. A tensor Φ will be called an affine tensor differential parameter of order (r, s) if its components

$$\Phi^{\alpha\dots\beta}_{\gamma\dots\delta}\left(L^{\lambda}_{\mu\nu};\ \dots;\ \frac{\partial^r L^{\lambda}_{\mu\nu}}{\partial x^{\xi}\dots\partial x^{\eta}};\ F^{(i)};\ \dots;\ \frac{\partial^s F^{(i)}}{\partial x^{\sigma}\dots\partial x^{\tau}}\right) \tag{15.1}$$

are functions of the L's and their derivatives to the rth order and of the scalar functions $F^1, \dots, F^w$ and their derivatives to the sth order, such that each component retains its form as a function of these quantities under the transformation (10.1).

We can likewise define tensor differential parameters for the metric space, the space of distant parallelism and the Weyl space, in an evident manner analogous to the definitions of the tensor differential invariants given in § 11.

Two simple scalar differential parameters of a metric space are those having components defined by (7)

$$\Delta_1\phi = g^{\alpha\beta}\frac{\partial\phi}{\partial x^{\alpha}}\frac{\partial\phi}{\partial x^{\beta}},$$

$$\Delta_1(\phi,\psi) = g^{\alpha\beta}\frac{\partial\phi}{\partial x^{\alpha}}\frac{\partial\psi}{\partial x^{\beta}};$$

these parameters are each of order (0, 1). There is also a simple scalar differential parameter of order (0, 2) of the metric space, with the component

$$\Delta_2\phi = g^{\alpha\beta}\phi_{,\alpha,\beta},$$

which was first constructed by Beltrami (8). For a general theory of the construction of scalar differential parameters the reader is referred to Chapter VII.

The quantities $\partial\phi/\partial x^{\alpha}$ are the covariant components of the vector which is normal to the surface S defined by the equation $\phi = 0$; in fact these quantities are the components of an affine vector differential parameter of order (0, 1). Hence it is evident that the vanishing of the differential parameter $\Delta_1\phi$, i.e. the equation

$$\Delta_1\phi = 0,$$

over the surface S, is a necessary and sufficient condition for the normals to this surface to form a null vector field.

If the fundamental quadratic differential form of a metric space is positive definite, then

$$\Delta_1(\phi,\psi) = 0$$

gives the condition that the surfaces $\phi = 0$ and $\psi = 0$ be normal at all points of intersection; by special definition let us moreover say that the above equation also gives the condition of normality of the surfaces $\phi = 0$ and $\psi = 0$ even if the fundamental quadratic differential form is not positive definite. Then it follows from $\Delta_1(x^{\alpha}, x^{\beta}) = g^{\alpha\beta}$, where $\alpha \neq \beta$, that a necessary and sufficient condition that the surfaces $x^{\alpha} = \text{const.}$ and $x^{\beta} = \text{const.}$ in any metric space be normal, is that $g^{\alpha\beta} = 0$ $(\alpha \neq \beta)$.

Again let ϕ^1 be an analytic function of the coordinates x^α of the region $\mathscr{R}$ covered by the coordinate system, such that

$$\Delta_1 \phi^1 \neq 0 \tag{15.2}$$

in $\mathscr{R}$. Then the differential equation

$$\Delta_1 (\phi^1, \psi) = 0 \tag{15.3}$$

admits $n-1$ independent analytic solutions $\phi^2, \ldots, \phi^n$ of the coordinates x^α, defined in an n-dimensional domain $\mathscr{D}$ of $\mathscr{R}$. Now in the domain $\mathscr{D}$ the Jacobian determinant

$$\begin{vmatrix} \dfrac{\partial \phi^1}{\partial x^1} & \cdots & \dfrac{\partial \phi^1}{\partial x^n} \\ \\ \\ \dfrac{\partial \phi^n}{\partial x^1} & & \dfrac{\partial \phi^n}{\partial x^n} \end{vmatrix}$$

cannot vanish identically; for suppose that such was the case so that ϕ^1 could be expressed as a function of $\phi^2, \ldots, \phi^n$. Then we would have

$$\Delta_1 \phi^1 = \sum_{\beta=2}^{n} \Delta_1 (\phi^1, \phi^\beta) \frac{\partial \phi^1}{\partial \phi^\beta},$$

and this would result in a contradiction on account of (15.2) and (15.3). Hence the functions $\phi^1, \ldots, \phi^n$ are independent and we can make the coordinate transformation $\bar{x}^\alpha = \phi^\alpha(x)$, where $\alpha = 1, \ldots, n$, of the coordinates of the domain $\mathscr{D}$ provided that $\mathscr{D}$ is sufficiently restricted; as a result of this transformation the above equations (15.3) give

$$\Delta_1 (\bar{x}^1, \bar{x}^\beta) = 0 \quad (\beta = 2, \ldots, n).$$

Hence $\bar{g}^{1\beta} = 0$ for $\beta = 2, \ldots, n$, and it follows that the fundamental quadratic differential form is given by

$$ds^2 = \bar{g}_{11} (d\bar{x}^1)^2 + \sum_{\mu=2}^{n} \sum_{\nu=2}^{n} \bar{g}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu,$$

in the $\bar{x}$ coordinate system defined through the domain $\mathscr{D}$.

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(4) The process of covariant differentiation was used by E. B. Christoffel, "Transformation der homogenen Differentialausdrücke zweiten Grades", *Journ. für reine und ange. Math.* **70** (1869), pp. 46–70. However, the name, covariant differentiation, and the discovery of its importance is due to G. Ricci, ref. (1), (4), **3** (1887), pp. 15–18.

(5) The Ricci identities appeared in the paper cited in ref. (4). See J. A. Schouten, *Der Ricci-Kalkül* (Julius Springer, 1924), p. 85 for the generalization to the affine case.

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(7) The name differential parameter and the notation are due to G. Lamé, *Leçons sur les coordinées curvilignes et leurs diverses applications* (Mallet-Bachelier, 1859), p. 5.

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CHAPTER III

PROJECTIVE INVARIANTS

16. Affine representation of projective spaces

The projective connection defined in § 4 is the analogue in the projective geometry of paths of the affine connection in the general affinely connected space (§ 2) or the affine geometry of paths (§ 3); this in fact is brought out very clearly on the basis of the point of view developed in § 81. We are consequently led to consider the equations of transformation of the components $\Pi^{\alpha}_{\beta\gamma}$ of the projective connection produced by coordinate transformations (1.2), i.e. transformations of the group $\mathfrak{G}$ defined in § 1. Making use of (9.5) and the corresponding equations of the type (13.1 (*b*)), we find in fact that[1]

$$\overline{\Pi}^{\sigma}_{\alpha\beta}\frac{\partial x^{\lambda}}{\partial \overline{x}^{\sigma}}=\frac{\partial^2 x^{\lambda}}{\partial \overline{x}^{\alpha}\partial \overline{x}^{\beta}}+\Pi^{\lambda}_{\mu\nu}\frac{\partial x^{\mu}}{\partial \overline{x}^{\alpha}}\frac{\partial x^{\nu}}{\partial \overline{x}^{\beta}}-\psi_{\alpha}\frac{\partial x^{\lambda}}{\partial \overline{x}^{\beta}}-\psi_{\beta}\frac{\partial x^{\lambda}}{\partial \overline{x}^{\alpha}}, \tag{16.1}$$

where

$$\psi_{\alpha}=\frac{1}{n+1}\frac{\partial \log (x\overline{x})}{\partial \overline{x}^{\alpha}},$$

and $(x\overline{x})$ has as before been used to denote the Jacobian determinant of the transformation of coordinates.* Use of the equations (16.1) will not, how-

* Corresponding to (16.1) we can deduce the equation of transformation of the projective parameter p defined in § 4. Under the arbitrary transformation (1.2) belonging to the group $\mathfrak{G}$ the projective parameter p becomes $\overline{p}$ and there exists a relation

$$p=f(\overline{p}) \tag{a}$$

between the parameters p and $\overline{p}$ along any particular path. In general the relation (*a*) will depend on the particular path and may contain certain arbitrary constants expressing the indeterminateness of the parameters.

Let us seek to find an explicit form of the relation (*a*). For this purpose we consider the equations (4.5) and the corresponding equations

$$\frac{d^2\overline{x}^{\alpha}}{d\overline{p}^2}+\overline{\Pi}^{\alpha}_{\beta\gamma}\frac{d\overline{x}^{\beta}}{d\overline{p}}\frac{d\overline{x}^{\gamma}}{d\overline{p}}=0 \tag{b}$$

in the $\overline{x}$ coordinate system. Eliminating the second derivatives in the above equations (*b*) by use of (1.2), the equations (4.5) and the relation (*a*), we may show that

$$\frac{\left[\overline{\Pi}^{\sigma}_{\alpha\beta}-\left(\Pi^{\lambda}_{\mu\nu}\frac{\partial x^{\mu}}{\partial \overline{x}^{\alpha}}\frac{\partial x^{\nu}}{\partial \overline{x}^{\beta}}+\frac{\partial^2 x^{\lambda}}{\partial \overline{x}^{\alpha}\partial \overline{x}^{\beta}}\right)\frac{\partial \overline{x}^{\sigma}}{\partial x^{\lambda}}\right]\frac{d\overline{x}^{\alpha}}{d\overline{p}}\frac{d\overline{x}^{\beta}}{d\overline{p}}}{\frac{d\overline{x}^{\sigma}}{d\overline{p}}}=\frac{\frac{d^2\overline{p}}{dp^2}}{\left(\frac{d\overline{p}}{dp}\right)^2}, \tag{c}$$

in which the bracket expression is equal to

$$-\frac{\delta^{\sigma}_{\alpha}}{n+1}\frac{\partial \log (x\overline{x})}{\partial \overline{x}^{\beta}}-\frac{\delta^{\sigma}_{\beta}}{n+1}\frac{\partial \log (x\overline{x})}{\partial \overline{x}^{\alpha}}$$

on account of (16.1). Consequently (*c*) becomes

$$\frac{d\log\left(\frac{d\overline{p}}{dp}\right)}{dp}=-\frac{2}{n+1}\frac{d\log (x\overline{x})}{dp};$$

hence

$$\frac{dp}{d\overline{p}}=\kappa(x\overline{x})^{N},\quad N=\frac{\;}{n+1}, \tag{d}$$

ever, lead to a theory of covariant differentiation or extension as in §§ 13 and 14 nor to an infinite sequence of tensor invariants analogous to the affine curvature tensor and its successive covariant derivatives. In fact such results depend evidently on a law of transformation of the precise type (9.5). We are consequently faced with the question of introducing invariants whose components possess more complicated laws of transformation than the components of a tensor* or else effecting a formal modification of the equations (16.1) which will result in a system of the type (9.5); this latter method is possible and will be adopted.

If we restrict ourselves to those coordinate transformations which satisfy the condition

$$(x\bar{x})=1, \tag{16.2}$$

equations (16.1) reduce to a system of equations of the type (9.5); in fact we have

$$\bar{\Pi}^{\lambda}_{\mu\nu}\frac{\partial x^{\alpha}}{\partial \bar{x}^{\lambda}}=\Pi^{\alpha}_{\beta\gamma}\frac{\partial x^{\beta}}{\partial \bar{x}^{\mu}}\frac{\partial x^{\gamma}}{\partial \bar{x}^{\nu}}+\frac{\partial^2 x^{\alpha}}{\partial \bar{x}^{\mu}\partial \bar{x}^{\nu}} \tag{16.3}$$

under the condition (16.2). This can be interpreted geometrically by saying that we consider transformations leaving volume unchanged. Thus the set of n contravariant vectors $\xi_{(1)}, \ldots, \xi_{(n)}$ with components $\xi^{\alpha}_{(\sigma)}$, where $\alpha, \sigma=1, \ldots, n$, define the volume $|\xi^{\alpha}_{(\sigma)}|$ and this volume is invariant under transformations satisfying (16.2). An arbitrary transformation (1.2) of the group $\mathfrak{G}$ satisfying (16.2) will be called an *equi-transformation* on account of the above property of invariance of volume. We may then speak of the *equi-projective* properties of the paths as those properties which are (1) independent of projective changes of the affine connection Γ and (2) invariant under equi-transformations. The body of theorems expressing equi-projective properties of the paths will constitute the *equi-projective geometry of paths*. Since (16.3) has the exact form of the transformation equations (9.5) we can introduce into the equi-projective geometry of paths a theory of covariant differentiation or extension analogous to that of the general affinely connected space; in addition we can construct an *equi-projective curvature tensor* and can form the infinite sequence of covariant derivatives of this tensor.

The idea of modifying the equations (16.1) so as to obtain a system of equations of the type (9.5) valid under arbitrary transformations (1.2) of the group $\mathfrak{G}$ suggests that the extraneous terms

$$-\frac{1}{n+1}\frac{\partial \log (x\bar{x})}{\partial \bar{x}^{\alpha}}\frac{\partial x^{\lambda}}{\partial \bar{x}^{\beta}}-\frac{1}{n+1}\frac{\partial \log (x\bar{x})}{\partial \bar{x}^{\beta}}\frac{\partial x^{\lambda}}{\partial \bar{x}^{\alpha}},$$

which appear in (16.1), be removed formally by the introduction of an additional coordinate—an idea in fact which is already to be found in classical projective geometry under the name of *homogeneous coordinates*. With this in mind let us construct from the transformations of the group $\mathfrak{G}$ a group of

where κ is an arbitrary constant and $(x\bar{x})$ is to be regarded as a function of $\bar{p}$ obtained by eliminating the variables $(\bar{x})$ occurring in $(x\bar{x})$ by means of the parametric equations of the path. Integrating (d) with the initial condition $p=0$ when $\bar{p}=0$, we have

$$p=\kappa\int_0^{\bar{p}}(x\bar{x})^N\,d\bar{p}$$

as the explicit form of (a).

* As an illustration of such a procedure the reader is referred to O. Veblen and J. M. Thomas, "Projective invariants of affine geometry of paths", *Ann. of Math.* (2), 27 (1926), pp. 279–96.

transformations $*\mathfrak{G}$ in a set of variables $x^0, x^1, \ldots, x^n$, any transformation of the group $*\mathfrak{G}$ being of the form(2)

$$\begin{cases} x^0 = \bar{x}^0 + \log(x\bar{x}) + \text{const.}, \\ x^i = f^i(\bar{x}^1, \ldots, \bar{x}^n); \end{cases} \tag{16.4}$$

more precisely we associate the above transformation (16.4) of the group $*\mathfrak{G}$ with the transformation (1.2) of the group $\mathfrak{G}$.

It will be found helpful before proceeding further to adopt the convention that Latin indices will take on values from $1, \ldots, n$ and Greek indices values from $0, 1, \ldots, n$. We shall adhere to this convention throughout the remainder of this chapter.

Let us now define a set of functions $*\Gamma^\alpha_{\beta\gamma}$ by means of the equations

$$\begin{aligned} *\Gamma^i_{jk} &= *\Gamma^i_{kj} = \Pi^i_{jk}, \\ *\Gamma^\alpha_{\beta 0} &= *\Gamma^\alpha_{0\beta} = -\frac{\delta^\alpha_\beta}{n+1}, \\ *\Gamma^0_{jk} &= *\Gamma^0_{kj} = \left(\frac{n+1}{n-1}\right)\mathfrak{B}^l_{jkl}, \end{aligned} \tag{16.5}$$

where $\mathfrak{B}^i_{jkl}$ denotes the components of the above mentioned equi-projective curvature tensor; we have

$$\mathfrak{B}^l_{jkl} = \frac{\partial \Pi^l_{jk}}{\partial x^l} - \Pi^i_{lj}\Pi^l_{ik} \tag{16.6}$$

in consequence of the fact that

$$\Pi^i_{ik} = \Pi^i_{ki} = 0.$$

Now the functions $*\Gamma$ transform according to the equations

$$*\bar{\Gamma}^\nu_{\beta\gamma}\frac{\partial x^\alpha}{\partial \bar{x}^\nu} = \frac{\partial^2 x^\alpha}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} + *\Gamma^\alpha_{\sigma\tau}\frac{\partial x^\sigma}{\partial \bar{x}^\beta}\frac{\partial x^\tau}{\partial \bar{x}^\gamma} \tag{16.7}$$

under the transformations of the group $*\mathfrak{G}$. In case the indices α, β, γ in the equations (16.7) have values $1, \ldots, n$, these equations are identical with the equations of the transformation of the projective connection given by (16.1). In case β or γ has the value 0, the equations (16.7) immediately reduce to an identity. The remaining case $\alpha = 0$, $\beta, \gamma \neq 0$ gives the equations of transformation of the components (16.6) of the contracted equi-projective curvature tensor under the group $\mathfrak{G}$.

Replacing the fundamental equations (16.1) by the equations (16.7), we can introduce into the projective geometry of paths a theory of covariant differentiation corresponding to that defined in § 13. We can also construct a *projective curvature tensor* with components

$$*B^\alpha_{\beta\gamma\delta} = \frac{\partial *\Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} - \frac{\partial *\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} + *\Gamma^\alpha_{\mu\delta}*\Gamma^\mu_{\beta\gamma} - *\Gamma^\alpha_{\mu\gamma}*\Gamma^\mu_{\beta\delta}, \tag{16.8}$$

subject to coordinate transformations of the group $^{\star}\mathfrak{G}$, and we can form the successive covariant derivatives of this curvature tensor so as to obtain a sequence analogous to (13.18).

Two points of view are now possible: Either we can adhere to the underlying n-dimensional space of the projective geometry of paths or we can associate with this space an $(n+1)$-dimensional affine space with coordinates $x^0, x^1, \ldots, x^n$ and components of affine connection $^{\star}\Gamma^{\alpha}_{\beta\gamma}$ subject to the transformations of the group $^{\star}\mathfrak{G}$. This latter viewpoint provides a direct $(n+1)$-*dimensional affine representation* $A^{\star}_{n+1}$ *of the projective space* P_n and lends itself very readily to geometrical interpretation.

17. Some geometrical interpretations

Some interesting interpretations relating to the $(n+1)$-dimensional representation $A^{\star}_{n+1}$ of the projective space P_n have been given by Whitehead(3). It is evident that the points of the projective space P_n can be put into one to one reciprocal correspondence with the curves of parameter x^0 of the affine representation $A^{\star}_{n+1}$. These latter curves will be called *rays*. Now the rays are paths in the affine representation $A^{\star}_{n+1}$, i.e. these curves satisfy the differential equations

$$\frac{d^2x^\alpha}{ds^2} + {}^{\star}\Gamma^{\alpha}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0 \tag{17.1}$$

for a suitable choice of the parameter s as can readily be verified (see footnote on p. 62).

Any hypersurface in $A^{\star}_{n+1}$ which is not generated by rays may be given the equation $\bar{x}^0 = 0$ by a suitable coordinate transformation (16.4) belonging to the group $^{\star}\mathfrak{G}$. Thus if

$$F(x^0, x^1, \ldots, x^n) = 0 \tag{17.2}$$

is the equation of a hypersurface, the derivative $\partial F/\partial x^0$ will vanish identically if, and only if, this surface is generated by rays. If this is not the case we can solve the equation (17.2) so as to obtain

$$x^0 + \phi(x^1, \ldots, x^n) = 0;$$

then a transformation of the form (16.4) in which the functions f^i are so selected that $\log(x\bar{x}) = \phi$ will lead to the above result.

If we write the equations (17.1) in the expanded form

$$\begin{cases} \dfrac{d^2x^i}{ds^2} + \Pi^i_{jk}\dfrac{dx^j}{ds}\dfrac{dx^k}{ds} - \dfrac{2}{n+1}\dfrac{dx^0}{ds}\dfrac{dx^i}{ds} = 0, \\[2ex] \dfrac{d^2x^0}{ds^2} + {}^{\star}\Gamma^0_{jk}\dfrac{dx^j}{ds}\dfrac{dx^k}{ds} - \dfrac{1}{n+1}\left(\dfrac{dx^0}{ds}\right)^2 = 0, \end{cases} \tag{17.3}$$

we see from the first set of these equations that the projection of any path, by means of the rays, on the hypersurface $x^0 = 0$, is one of the paths defined on this surface by the components Π^i_{jk} of the projective connection. Now let C be any path in $A^{\star}_{n+1}$ which is not a ray, and let S_n be a hypersurface which contains C and which is not generated by rays. We may assume without loss of generality in consequence of the above result that $x^0 = 0$ is the equation of S_n. Let Σ_2 be the two dimensional surface generated by the rays which intersect C, let P and Q be any two points on Σ_2 not on the same ray, and let PQ denote the path joining these points; it is to be assumed that we are dealing with a region in which one and only one path passes through any two points (see footnote on p. 6). We shall show that PQ is contained in Σ_2.

Let the rays through P and Q meet C in P_0 and Q_0 respectively. See Fig. 7. Denote by $P_0 Q_0$ the projection of PQ on S_n; then $P_0 Q_0$ is the path of S_n joining P_0 and Q_0 which is defined by the components Π^i_{jk} of the projective connection. But from (17.3) it follows that C is a similar path of S_n and since it joins P_0 and Q_0 the paths C and $P_0 Q_0$ must coincide. Hence the path PQ lies on a surface generated by rays which pass through the path C, i.e. the path PQ lies on the surface Σ_2. The result obtained may be stated by saying that the *surface generated by the rays which intersect any path of $A^{\star}_{n+1}$ is a plane.**

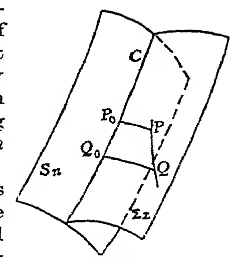

Fig. 7.

It follows at once that if a plane p-space exists which does not contain a ray, then the rays which meet it generate a plane $(p+1)$-space. Plane p-spaces in P_n are therefore represented by plane $(p+1)$-spaces in $A^{\star}_{n+1}$ and, in particular, paths by planes.

18. Projective tensors and invariants

Definition of Projective Tensor. A set of functions $^{\star}T^{\alpha\ldots\beta}_{\gamma\ldots\delta}(x)$ of the coordinates $x^1, \ldots, x^n$ of the region $\mathscr{R}$, where $\alpha, \ldots, \beta, \gamma, \ldots, \delta = 0, 1, \ldots, n$, constitute the components of a relative projective tensor $^{\star}T$ of weight W with respect to the x coordinate system of the affine representation $A^{\star}_{n+1}$, provided that the functions $^{\star}T^{\alpha\ldots\beta}_{\gamma\ldots\delta}$ transform according to the equations

$$^{\star}\overline{T}^{\mu\ldots\nu}_{\sigma\ldots\tau}(\overline{x}) = (x\overline{x})^W \, {^{\star}T}^{\alpha\ldots\beta}_{\gamma\ldots\delta}(x) \frac{\partial \overline{x}^\mu}{\partial x^\alpha} \cdots \frac{\partial \overline{x}^\nu}{\partial x^\beta} \frac{\partial x^\gamma}{\partial \overline{x}^\sigma} \cdots \frac{\partial x^\delta}{\partial \overline{x}^\tau} \tag{18.1}$$

when the coordinates $x^0, x^1, \ldots, x^n$ are transformed by (16.4). *The object obtained by abstraction from the above components with respect to the totality of coordinate systems whose coordinates are related by transformations of the group $^{\star}\mathfrak{G}$ is called the tensor $^{\star}T$.*

It is to be observed in particular that while the indices on the symbol of the components of the tensor $^{\star}T$ assume the range of values $0, 1, \ldots, n$ that these components depend only on the coordinates $x^1, \ldots, x^n$ of the region $\mathscr{R}$. This requirement further indicates the essentially fictitious nature of the coordinate x^0 which, as we saw in § 16, was introduced solely for the purpose of securing invariants of formal tensor character.

We see that the theory of covariant differentiation already mentioned in § 16, leads to covariant derivatives of a projective tensor $^{\star}T$ which are likewise projective tensors in the sense of the above definition.

Projective tensor differential invariants of order r are to be defined in a manner analogous to the definition of the affine tensor differential invariant in § 11 by replacing, in the statement of this latter definition, the components L by the components $^{\star}\Gamma$ and the transformations (10.1) and (1.2) by the transformation (18.1) and (16.4), respectively. Also an analogous extension

* A subspace is described as plane if any path which meets it twice is contained in it. In particular a plane two dimensional surface will be called a plane. Cf. É. Cartan, *Leçons sur la Géométrie des Espaces de Riemann* (Gauthier-Villars, 1928), Chapter V.

of the definition of the affine tensor differential parameter defined in § 15 gives us the projective tensor differential parameter.

The projective curvature tensor with components $^{\star}B^{\alpha}_{\beta\gamma\delta}$ defined by (16.8) is an example of a projective tensor differential invariant.

If either β, γ or δ has the value 0 the components $^{\star}B^{\alpha}_{\beta\gamma\delta}$ vanish identically, i.e.

$$^{\star}B^{\alpha}_{0\gamma\delta}={}^{\star}B^{\alpha}_{\beta0\delta}={}^{\star}B^{\alpha}_{\beta\gamma0}=0, \tag{18.2}$$

as can be seen from (16.5) and (16.8). Hence the quantities defined by

$$W^{i}_{jkl}={}^{\star}B^{i}_{jkl}$$

constitute the components of an affine tensor in the sense of § 10, i.e.

$$\overline{W}^{i}_{jkl}\frac{\partial x^{a}}{\partial\bar{x}^{i}}=W^{a}_{bcd}\frac{\partial x^{b}}{\partial\bar{x}^{j}}\frac{\partial x^{c}}{\partial\bar{x}^{k}}\frac{\partial x^{d}}{\partial\bar{x}^{l}}; \tag{18.3}$$

in the derivation of these equations use is made of the fact that the derivatives $\partial x^{a}/\partial\bar{x}^{0}$, for $a=1, \ldots, n$, vanish in consequence of (16.4). Written more explicitly we have(4)

$$W^{i}_{jkl}=\mathfrak{B}^{i}_{jkl}+\frac{\delta^{i}_{k}}{n-1}\mathfrak{B}^{m}_{jlm}-\frac{\delta^{i}_{l}}{n-1}\mathfrak{B}^{m}_{jkm},$$

or

$$W^{i}_{jkl}=B^{i}_{jkl}+\frac{1}{n+1}\delta^{i}_{j}(B_{kl}-B_{lk}) \tag{18.4}$$
$$-\frac{n}{n^{2}-1}(\delta^{i}_{l}B_{jk}-\delta^{i}_{k}B_{jl})-\frac{1}{n^{2}-1}(\delta^{i}_{l}B_{kj}-\delta^{i}_{k}B_{lj}),$$

as can be deduced immediately from (16.5) and (16.8). It is interesting to observe in consequence of (18.4) that the contracted components W^{i}_{jki} vanish identically; hence the quantities W^{i}_{jil} likewise vanish identically owing to the skew-symmetric character of the indices k and l in the components W^{i}_{jkl}. Moreover it can be shown that the identical relations $W^{i}_{ikl}=0$ are satisfied.*

By evaluating (16.8) we find that

$$^{\star}B^{0}_{jkl}=-\Gamma^{i}_{im}W^{m}_{jkl}+\frac{n}{n-1}(B_{jk,l}-B_{jl,k})+\frac{1}{n-1}(B_{kj,l}-B_{lj,k}), \tag{18.5}$$

where B_{jk} are the components B^{l}_{jkl} of the contracted affine curvature tensor. By covariant differentiation of the expression (18.4) for W^{i}_{jkl} and by contraction on the index of differentiation, we find

$$W^{i}_{jkl,i}=\frac{1}{n+1}(B_{kl,j}-B_{lk,j})-\frac{1}{n^{2}-1}(B_{kj,l}-B_{lj,k})+\frac{n^{2}-n-1}{n^{2}-1}(B_{jk,l}-B_{jl,k})$$
$$=\left(\frac{n-2}{n+1}\right)\left\{\frac{n}{n-1}(B_{jk,l}-B_{jl,k})+\frac{1}{n-1}(B_{kj,l}-B_{lj,k})\right\},$$

on making use of the identities (49.13) and (49.14) to be derived later. Hence the above expression for $^{\star}B^{0}_{jkl}$ can also be written in the form

$$^{\star}B^{0}_{jkl}=\left(\frac{n+1}{n-2}\right)W^{i}_{jkl,i}-\Gamma^{i}_{im}W^{m}_{jkl}\quad(n>2). \tag{18.6}$$

By expanding the right members of (18.4) in terms of the Π^{i}_{jk} and their derivatives, it is seen that for $n=2$ the components W^{i}_{jkl} vanish identically. Hence for $n=2$, equations (18.5) become

$$^{\star}B^{0}_{jkl}=2(B_{jk,l}-B_{jl,k})+(B_{kj,l}-B_{lj,k})\quad(n=2). \tag{18.7}$$

* This follows from identities of the type (51.6) and the fact that the components W^{i}_{jki} and W^{i}_{jil} vanish identically.

Hence the quantities $^{\star}B^0_{jkl}$ constitute the components of an affine tensor, i.e.

$$(18.8)\qquad {}^{\star}\bar{B}^0_{jkl} = {}^{\star}B^0_{abc}\frac{\partial x^a}{\partial \bar{x}^j}\frac{\partial x^b}{\partial \bar{x}^k}\frac{\partial x^c}{\partial \bar{x}^l},$$

where the values 1, 2 only are assumed by all indices in these equations.

In order to indicate the projective character of the tensor whose components transform by (18.3) as well as the relationship of this tensor to the projective curvature tensor, we may refer to it as the *projective-affine curvature tensor*. A similar designation will be applied to the tensor with components transforming by (18.8) under the conditions that the projective space P_n is two dimensional.

19. Transformations of the group $^{\star}\mathfrak{G}$

Consider an $(n+1)$-dimensional representation $\tilde{A}^{\star}_{n+1}$ of a projective space $\tilde{P}_n$, possessing a connection with components $^{\star}\tilde{\Gamma}^{\alpha}_{\beta\gamma}(\tilde{x})$ defined by equations of the type (16.5). Form the equations

$$(19.1a)\qquad {}^{\star}\Gamma^{\nu}_{\beta\gamma}(x)\frac{\partial \tilde{x}^{\alpha}}{\partial x^{\nu}} = \frac{\partial^2 \tilde{x}^{\alpha}}{\partial x^{\beta}\partial x^{\gamma}} + {}^{\star}\tilde{\Gamma}^{\alpha}_{\mu\nu}(\tilde{x})\frac{\partial \tilde{x}^{\mu}}{\partial x^{\beta}}\frac{\partial \tilde{x}^{\nu}}{\partial x^{\gamma}},$$

involving the components $^{\star}\Gamma^{\alpha}_{\beta\gamma}(x)$ of the representation $A^{\star}_{n+1}$. Suppose that (19.1a) admits a solution

$$(19.2)\qquad \tilde{x}^{\alpha} = \phi^{\alpha}(x^0, x^1, \ldots, x^n)$$

which establishes a one to one reciprocal correspondence between the points of the two representations $A^{\star}_{n+1}$ and $\tilde{A}^{\star}_{n+1}$. It will be assumed that the functions ϕ^{α} are analytic; hence the functions analogous to the ϕ^{α} which occur in the relation inverse to (19.2) will likewise be analytic. The question then arises if the relation (19.2) belongs necessarily to the group $^{\star}\mathfrak{G}$ (5).

To consider this question we derive several particular sets of equations from (19.1a) or the equivalent equations

$$(19.1b)\qquad {}^{\star}\tilde{\Gamma}^{\nu}_{\beta\gamma}(\tilde{x})\frac{\partial x^{\alpha}}{\partial \tilde{x}^{\nu}} = \frac{\partial^2 x^{\alpha}}{\partial \tilde{x}^{\beta}\partial \tilde{x}^{\gamma}} + {}^{\star}\Gamma^{\alpha}_{\sigma\tau}(x)\frac{\partial x^{\sigma}}{\partial \tilde{x}^{\beta}}\frac{\partial x^{\tau}}{\partial \tilde{x}^{\gamma}},$$

$$(19.1c)\qquad {}^{\star}\tilde{\Gamma}^{\alpha}_{\beta\gamma}(\tilde{x}) = \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\sigma}}\left(\frac{\partial^2 x^{\sigma}}{\partial \tilde{x}^{\beta}\partial \tilde{x}^{\gamma}} + {}^{\star}\Gamma^{\sigma}_{\mu\nu}(x)\frac{\partial x^{\mu}}{\partial \tilde{x}^{\beta}}\frac{\partial x^{\nu}}{\partial \tilde{x}^{\gamma}}\right).$$

If we put $\gamma = 0$ in (19.1b), we obtain

$$(19.3)\qquad \frac{\partial^2 x^{\alpha}}{\partial \tilde{x}^{\beta}\partial \tilde{x}^0} = -\frac{1}{n+1}\frac{\partial x^{\alpha}}{\partial \tilde{x}^{\beta}} - {}^{\star}\Gamma^{\alpha}_{\sigma\tau}(x)\frac{\partial x^{\sigma}}{\partial \tilde{x}^{\beta}}\frac{\partial x^{\tau}}{\partial \tilde{x}^0}.$$

Also putting $\alpha = \gamma$ and summing in (19.1c), we obtain

$$(19.4)\qquad \frac{\partial x^0}{\partial \tilde{x}^{\beta}} = \delta^0_{\beta} + \frac{\partial \log |x\tilde{x}|}{\partial \tilde{x}^{\beta}}$$

where

$$|x\tilde{x}| = \begin{vmatrix} \dfrac{\partial x^0}{\partial \tilde{x}^0} & \dfrac{\partial x^0}{\partial \tilde{x}^1} & \cdots & \dfrac{\partial x^0}{\partial \tilde{x}^n} \\ \dfrac{\partial x^1}{\partial \tilde{x}^0} & \dfrac{\partial x^1}{\partial \tilde{x}^1} & \cdots & \dfrac{\partial x^1}{\partial \tilde{x}^n} \\ \\ \dfrac{\partial x^n}{\partial \tilde{x}^0} & \dfrac{\partial x^n}{\partial \tilde{x}^1} & \cdots & \dfrac{\partial x^n}{\partial \tilde{x}^n} \end{vmatrix}$$

Now we can deduce from (19.1*b*) the equations

(19.5)
$$\star\tilde{B}^{\sigma}_{\beta\gamma\delta}\frac{\partial x^\alpha}{\partial \tilde{x}^\sigma} = \star B^{\alpha}_{\mu\nu\eta}\frac{\partial x^\mu}{\partial \tilde{x}^\beta}\frac{\partial x^\nu}{\partial \tilde{x}^\gamma}\frac{\partial x^\eta}{\partial \tilde{x}^\delta},$$

in which the derivatives are calculated from the transformation (19.2). Putting $\delta = 0$ in (19.5), we have in consequence of (18.2) that

(19.6)
$$\star B^{\alpha}_{\mu\nu\eta}\frac{\partial x^\eta}{\partial \tilde{x}^0} = 0.$$

If we put

(19.7)
$$\xi^\alpha = \frac{\partial x^\alpha}{\partial \tilde{x}^0},$$

equations (19.3) and (19.6) go over into the following two sets of equations

(19.8*a*)
$$\xi^\alpha_{,\beta} = -\frac{\delta^\alpha_\beta}{n+1},$$

and

(19.8*b*)
$$\star B^{\alpha}_{\beta\gamma\delta}\xi^\delta = 0,$$

respectively; here the ξ^α are to be considered as functions of the coordinates $x^0, x^1, \ldots, x^n$ resulting from (19.7) by means of the relation inverse to (19.2), and

$$\xi^\alpha_{,\beta} = \frac{\partial \xi^\alpha}{\partial x^\beta} + \star\Gamma^\alpha_{\sigma\beta}\xi^\sigma,$$

i.e. these quantities constitute the components of the covariant derivative of the contravariant vector ξ. It is evident that the vector components $\tilde{\xi}^\alpha(\tilde{x})$, which are determined from the components ξ^α by (19.2) and the vector law of transformation, must satisfy equations analogous to (19.8) on account of the tensor character of these equations.

Now it can happen that the equations (19.8) possess only one independent solution $\xi^\alpha(x)$ or that these equations possess two or more linearly independent solutions. These two cases will be referred to as hypothesis α and hypothesis β respectively. We proceed to examine these two hypotheses in turn.

Hypothesis α. It is seen that $\xi^\alpha = \delta^\alpha_0$ constitutes a solution of the equations (19.8) and under the present hypothesis there exists no other linearly in-

dependent solution. The same must therefore be true of the equations analogous to (19.8) in the coordinates $\tilde{x}^\alpha$; hence the components $\xi^\alpha(x)$, i.e. δ_0^α, must be transformed by (19.2) and the contravariant vector law of transformation into components $\tilde{\xi}^\alpha(\tilde{x})$ which are of the form $\delta_0^\alpha \psi(\tilde{x}^0, \dots, \tilde{x}^n)$, where ψ is an analytic function of the coordinates $\tilde{x}^\alpha$. Since the components $\delta_0^\alpha \psi(\tilde{x})$ must satisfy (19.8a) taken with respect to the coordinates $\tilde{x}^\alpha$, we have

$$\delta_0^\alpha \frac{\partial \psi}{\partial \tilde{x}^\beta} = \frac{\delta_\beta^\alpha}{n+1}(\psi - 1). \tag{19.9}$$

Putting $\alpha = \beta \neq 0$ in (19.9) we obtain $\psi = 1$. In other words the components δ_0^α are transformed directly into the same components δ_0^α by (19.2) and the contravariant vector law of transformation. This gives

$$\frac{\partial x^\alpha}{\partial \tilde{x}^0} = \delta_0^\alpha;$$

hence (19.2) is equivalent to a set of equations of the form

$$x^0 = \tilde{x}^0 + f^0(\tilde{x}^1, \dots, \tilde{x}^n), \quad x^i = f^i(\tilde{x}^1, \dots, \tilde{x}^n),$$

and the determinant $|x\tilde{x}|$ is equal to $(x\tilde{x})$, i.e. the Jacobian determinant formed from the above n functions f^i. Equations (19.4) for $\beta \neq 0$ then show that

$$f^0 = \log(x\tilde{x}) + \text{const.}$$

Hence we have the following

THEOREM A. *Under hypothesis* α *the relation* (19·2) *belongs to the group* $^*\mathfrak{G}$.

Hypothesis β. Let us suppose that $\xi^\alpha(x)$ and $\eta^\alpha(x)$ are two linearly independent analytic solutions of the equations (19.8); we shall suppose moreover that the ξ^α are identical with the constants δ_0^α since these latter quantities constitute a solution of (19.8).* Now from the fact that η^α satisfies (19.8a) we have

$$\frac{\partial \eta^\alpha}{\partial x^0} - \frac{\eta^\alpha}{n+1} = -\frac{\delta_0^\alpha}{n+1};$$

hence

$$\eta^\alpha(x) = \delta_0^\alpha + e^{x^0/(n+1)} \phi_1^\alpha(x^1, \dots, x^n). \tag{19.10}$$

It is evident therefore that the $\eta^\alpha(x)$ can be considered without essential loss of generality to be defined throughout the entire representation A_{n+1}^*; in fact if the above functions ϕ_1^α were not defined throughout the n-dimensional region $\mathscr{R}$ we could restrict the region $\mathscr{R}$ so as to secure this result (see §1).

Let us now form the equations

$$\frac{d\bar{x}^\alpha}{dt} = e^{\bar{x}^0/(n+1)} \phi_1^\alpha(\bar{x}^1, \dots, \bar{x}^n), \tag{19.11}$$

* If the components $\xi^\alpha = \delta_0^\alpha$ and η^α were not independent, we would have $\eta^\alpha = \delta_0^\alpha \psi(x)$, and we could then deduce $\psi = 1$ by the above method, i.e. the components ξ^α and η^α would be identical.

where the variables $\bar{x}^\alpha$ have been used in place of x^α as a mere matter of notation. It is then easily seen that

$$\frac{d^p \bar{x}^\alpha}{dt^p} = e^{p\bar{x}^0/(n+1)} \phi_p^\alpha (\bar{x}^1, \ldots, \bar{x}^n),$$

in which the functions $\phi_p^\alpha (\bar{x})$ are analytic throughout the region $\mathscr{R}$. Denoting by x^α the coordinates of an arbitrary point P of the representation $A^\star_{n+1}$, the set of solutions $\bar{x}^\alpha$ of (19.11) which reduce to x^α for $t=0$ may therefore be expanded in the power series

$$(19.12) \qquad \bar{x}^\alpha = x^\alpha + \sum_{p=1}^{\infty} \frac{[e^{x^0/(n+1)} t]^p}{p!} \phi_p^\alpha (x^1, \ldots, x^n).$$

Hence if t_1 is any positive number, it is evident that we can find an $(n+1)$-dimensional region $\mathscr{R}^\star$ of $A^\star_{n+1}$ defined by $| x^0 | \geqq k$, where x^0 is negative and k is a sufficiently large positive number, such that the series (19.12) will converge for $| t | \leqq t_1$ and for all points P of the region $\mathscr{R}^\star$.

Now denote by ϵ a positive constant which may be taken as small as we please; then it is seen that the differences $\bar{x}^\alpha - x^\alpha$ determined by (19.12) will always be such that $| \bar{x}^\alpha - x^\alpha | < \epsilon$ for a suitable choice of the above constant k. Assuming that such a choice of the constant k has been made, we see that there must therefore exist an $(n+1)$-dimensional region $\mathscr{R}^{\star\star}$ contained in $\mathscr{R}^\star$ such that (19.12) regarded as a *point transformation* will displace points of $\mathscr{R}^{\star\star}$ into points of $\mathscr{R}^\star$. It is likewise evident that the region $\mathscr{R}^{\star\star}$ can be so restricted that the functional determinant of (19.12) will be different from zero for any point of this region, and in fact so that (19.12) will possess a single valued inverse throughout $\mathscr{R}^{\star\star}$; it will be supposed in the following that the region $\mathscr{R}^{\star\star}$ is of this character.

It can be shown that the components ${}^\star\Gamma^\alpha_{\beta\gamma}(x)$ are transformed by (16.7) and the above point transformation (19.12) into components ${}^\star\Gamma^\alpha_{\beta\gamma}(\bar{x})$, i.e. the quantities ${}^\star\Gamma^\alpha_{\beta\gamma}(x)$ and ${}^\star\Gamma^\alpha_{\beta\gamma}(\bar{x})$ are the *same* functions of the coordinates x^α and $\bar{x}^\alpha$, respectively.*

* We observe that the right member of (19.11) is equal to $\eta^\alpha - \delta^\alpha_0$, and that we have

$$(\eta^\alpha - \delta^\alpha_0)_{,\beta,\gamma} = 0;$$

hence

$$(\eta^\alpha - \delta^\alpha_0)_{,\beta,\gamma} + {}^\star B^\alpha_{\beta\gamma\lambda} (\eta^\lambda - \delta^\lambda_0) = 0.$$

These latter equations constitute the condition for the components ${}^\star\Gamma^\alpha_{\beta\gamma}(x)$ to retain their form as functions of the coordinates under the transformations (16.7) and (19.12). Cf. L. P. Eisenhart, *Non-Riemannian Geometry, Amer. Math. Soc.* (1927), p. 126. Equations (46.9) of *Non-Riemannian Geometry* are seen to be identical with the above conditions when account is taken of slight differences of notation.

We may observe likewise in this connection that if $\xi^\alpha(x)$ is a solution of (19.8), the congruence of the curves defined by

$$\frac{dx^\alpha}{ds} = \xi^\alpha(x)$$

are paths in the representation $A^\star_{n+1}$. Thus we have

$$\frac{d^2 x^\alpha}{ds^2} + {}^\star\Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} + \frac{1}{n+1}\frac{dx^\alpha}{ds} = \xi^\alpha_{,\beta}\xi^\beta + \frac{\xi^\alpha}{n+1} = 0$$

by (19.8 a). The above statement then follows by the results of § 3. In particular, since $\xi^\alpha = \delta^\alpha_0$ is a solution of (19.8), we see that the rays defined in § 17 are paths.

Now consider the transformations, analogous to (19.12), which are determined by the equations

$$\frac{dx^{\alpha}}{dt} = X^{\alpha}(\bar{x}), \tag{19.13}$$

and the initial conditions $\bar{x}^{\alpha} = x^{\alpha}$ for $t = 0$; these transformations carry the components $V^{\alpha}(x)$ of a contravariant vector into the components $\overline{V}^{\alpha}(\bar{x})$ such that

$$\overline{V}^{\alpha}(\quad = V^{\alpha}(\bar{x}) + \sum_{p=1}^{\infty} \frac{(-t)^p}{p!} V^{\alpha}_{p}, \tag{19.14}$$

where t is the value of the parameter which determines a particular transformation of the set, and the V_p are functions of the coordinates $\bar{x}^{\alpha}$ given by*

$$\begin{aligned} V^{\alpha}_{1} &= \frac{\partial V^{\alpha}(\bar{x})}{\partial \bar{x}^{\beta}} X^{\beta}(\bar{x}) - \frac{\partial X^{\alpha}(\bar{x})}{\partial \bar{x}^{\beta}} V^{\beta}(\bar{x}), \\ V^{\alpha}_{p+1} &= \frac{\partial V^{\alpha}_{p}(\bar{x})}{\partial \bar{x}^{\beta}} X^{\beta}(\bar{x}) - \frac{\partial X^{\alpha}(\bar{x})}{\partial \bar{x}^{\beta}} V^{\beta}_{p}(\bar{x}). \end{aligned} \tag{19.15}$$

Now take $\eta^{\alpha} - \xi^{\alpha}$ and ξ^{α} for X^{α} and V^{α}, respectively. Then we shall have

$$\xi^{\alpha}_{1} = \frac{X^{\alpha}}{n+1},$$

use being made of (19.8*a*); also

$$\xi^{\alpha}_{p} = 0 \quad \text{if} \quad p > 1.$$

Hence from (19.14) we obtain

$$\bar{\xi}^{\alpha}(\bar{x}) = \xi^{\alpha}(\bar{x}) + \frac{t}{n+1}[\eta^{\alpha}(\bar{x}) - \xi^{\alpha}(\bar{x})],$$

where ξ^{α} stands for the value δ^{α}_{0}. Taking $t = (n+1)$, we have

$$\begin{aligned} {}^{\star}\Gamma^{\alpha}_{\beta\gamma}(x) &\to {}^{\star}\Gamma^{\alpha}_{\beta\gamma}(\bar{x}), \\ \xi^{\alpha}(x) = \delta^{\alpha}_{0} &\to \eta^{\alpha}(\bar{x}), \end{aligned}$$

i.e. *the point transformation* (19.12) *for* $t = (n+1)$ *sends the components* ${}^{\star}\Gamma^{\alpha}_{\beta\gamma}(x)$ *and* $\xi^{\alpha}(x) = \delta^{\alpha}_{0}$ *for points P of the region* $\mathscr{R}^{\star\star}$ *into the components*

* The proof of these equations is immediate. Let us take a coordinate system in which the contravariant vector components X^{α} are δ^{α}_{0} (see § 10). Then the set of transformations determined by (19.13) becomes $\bar{x}^0 = x^0 + t$, $\bar{x}^i = x^i$ ($i = 1, \ldots, n$), and we have

$$\begin{aligned} \overline{V}^{\alpha}(\bar{x}) &= V^{\alpha}(\bar{x}^0 - t, \bar{x}^1, \ldots, \bar{x}^n) \\ &= V^{\alpha}(\bar{x}) + \sum_{p=1}^{\infty} \frac{(-t)^p}{p!} V^{\alpha}_{p}, \end{aligned}$$

where

$$V^{\alpha}_{p} = \frac{\partial^p V^{\alpha}(\bar{x})}{\partial \bar{x}^0 \ldots \partial \bar{x}^0}. \tag{a}$$

Now it is easily shown that the right members of (19.15) are the components of contravariant vectors. Hence (19.14) and (19.15) are invariant in form under transformations of coordinates, and since (19.15) reduces to the above equations (*a*) in the coordinate system for which $X^{\alpha} = \delta^{\alpha}_{0}$, it follows that (19.14) and (19.15) are valid in general.

$^*\Gamma^{\alpha}_{\beta\gamma}(\bar{x})$ *and* $\eta^{\alpha}(\bar{x})$, *respectively, where the* $\bar{x}^{\alpha}$ *are coordinates of points of the region* $\mathscr{R}^*$.*

Let us now denote by T the relation (19.2) and by S the point transformation (19.12) in which the x^{α} are the coordinates of a point P of the region $\mathscr{R}^{**}$. Representing S by $\bar{x}^{\alpha} = \lambda^{\alpha}(x)$, the relation TS defined by

$$\tilde{x}^{\alpha} = \phi^{\alpha}[\lambda^0(x), \ldots, \lambda^n(x)],$$

in which the x^{α} are the coordinates of an arbitrary point P of the region $\mathscr{R}^{**}$, is such that (1) the functions ϕ^{α} are analytic, (2) the functional determinant of the ϕ^{α} is everywhere different from zero, and (3) these equations have a unique inverse.

Now suppose that the quantities δ^{α}_0, considered as the components of a contravariant vector in the $\tilde{x}$ coordinate system, are carried by T into the components $\eta^{\alpha}(x)$ with respect to the x coordinate system; then the quantities $\eta^{\alpha}(x)$ will satisfy the equations (19.8). By the above italicized result it therefore follows that TS will transform the contravariant vector components δ^{α}_0 into the components δ^{α}_0 and the components $^*\Gamma^{\alpha}_{\beta\gamma}(x)$ into the components $^*\bar{\Gamma}^{\alpha}_{\beta\gamma}(\tilde{x})$, i.e. the relation TS satisfies the equations (19.1 a). By the argument given under hypothesis α the relation TS therefore has the form (16.4). In conformity with this result, let us now replace the original n-dimensional region $\mathscr{R}$ by that portion of this region which corresponds to the $(n+1)$-dimensional region $\mathscr{R}^{**}$ and likewise the relation T by the relation TS; this is a formality of the sort which we used above and which involves no essential loss of generality. We can then state the following

THEOREM B. *Under hypothesis* β *an analytic solution of* (19·1 a) *of the form* (19.2) *exists which belongs to the group* $^*\mathfrak{G}$ *and which defines a one to one reciprocal correspondence between the representations* A^*_{n+1} *and* $\tilde{A}^*_{n+1}$.

A simple illustration of a case for which there will exist more than one independent solution ξ^{α} of the equations (19.8) is obtained by taking all the components Π^i_{jk} of the projective connection equal to zero.

The above Theorems A and B have an important application in the problem of the equivalence of projective spaces in § 88.

REFERENCES

(1) The idea of a projective connection is due to É. Cartan who defined the connection by means of the differential equations relating the flat projective tangent spaces associated with each point of the manifold. See "Sur les variétés à connexion projective", *Bull. Soc. Math. de France*, 52 (1924), pp. 205–41. Cartan's ideas were further developed by J. A. Schouten with the aid of König connections. See "On the place of conformal and projective geometry in the theory of linear displacement", *Proc. Kon. Akad. Amsterdam*, 27 (1924), pp. 405–29. The explicit definition of the projective connection $\Pi^{\alpha}_{\beta\gamma}$ as defined in § 4, the projective parameter and the equi-projective geometry of paths in § 16 was given by T. Y. Thomas, ref. (7), Chapter I.

* This result is seen immediately to be valid in case the components η^{α} are dependent on the components $\xi^{\alpha} = \delta^{\alpha}_0$, since then $\eta^{\alpha} = \delta^{\alpha}_0$ by the remark in the preceding footnote and the transformation (19.12) reduces to the identity.

(2) T. Y. Thomas, "A projective theory of affinely connected manifolds", *Math. Zeit.* **25** (1926), pp. 723–33. The application of the results of this paper to the scheme of Cartan was discussed by H. Weyl, "On the foundations of general infinitesimal geometry", *Bull. Amer. Math. Soc.* **35** (1929), pp. 716–25.

(3) J. H. C. Whitehead, "The representation of projective spaces", *Ann. of Math.* (2), **32** (1931), pp. 327–60.

(4) The affine tensor whose components are defined by (18.4) is identical with a tensor originally introduced by H. Weyl and called the Weyl projective curvature tensor by most writers. He also defined the projective-affine curvature tensor for the two dimensional projective space P_2. See H. Weyl, ref. (6), Chapter I. Also cf. J. M. Thomas, "Note on the projective geometry of paths", *Proc. N.A.S.* **11** (1925), pp. 207–9.

(5) This question was originally raised by T. Y. Thomas, "Concerning the group $^{\star}\mathfrak{G}$ of transformations", *Proc. N.A.S.* **14** (1928), pp. 728–34; certain of the results of this paper are, however, invalid owing to the appearance of an incorrect algebraic sign in the last set of equations (5.3). In this paper it is shown that (19.2) belongs to the group $^{\star}\mathfrak{G}$ under the condition that $\partial \bar{x}^{\alpha}/\partial x^{0} = \delta^{\alpha}_{0}$ at some point P of the representation $^{\star}A_{n+1}$. This condition was later removed by J. H. C. Whitehead, ref. (3). The method used in § 19 is essentially that given by Whitehead.

CHAPTER IV

CONFORMAL INVARIANTS

20. Fundamental conformal-affine tensor

Since the components $g_{\alpha\beta}$ of the fundamental quadratic differential form are determined in a conformal space only to within an arbitrary factor $\sigma(x)$ in accordance with equations (7.1), we see that the general equations of transformation of these components, corresponding to arbitrary coordinate transformations of the group $\mathfrak{G}$, must have the form

$$\bar{g}_{\mu\nu}=\sigma(x)\, g_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu}.$$

Taking the determinants of the quantities in the left and right members of these equations, we have

$$|\bar{g}_{\mu\nu}|=\sigma^n\,|g_{\alpha\beta}|\,(x\bar{x})^2.$$

Hence

$$\bar{G}_{\mu\nu}=(x\bar{x})^{-2/n}\, G_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu}, \tag{20.1}$$

where

$$G_{\alpha\beta}=\frac{g_{\alpha\beta}}{(\pm\,|g_{\sigma\tau}|)^{1/n}}.$$

It will be assumed that the negative sign is to be taken in case $|g_{\sigma\tau}|<0$ in the region $\mathscr{R}$ and n is even, so as to avoid the appearance of the imaginary quantity which would otherwise occur; in other cases the positive sign will be employed.

The above quantities $G_{\alpha\beta}$ constitute the components of a symmetric tensor G which is an affine tensor in the general sense of § 10; we observe that the condition $|G_{\alpha\beta}|=\pm 1$ is identically satisfied. The conformal character of this tensor is seen from the fact that its components are unaltered by the conformal transformation (7.1). Since the tensor G plays an analogous role in the theory of the conformal space to that played in the theory of the metric space by the fundamental metric tensor, we shall refer to it as the *fundamental conformal-affine tensor* (1).

Suppose that we replace the equations (20.1) by the equations of transformation of the components $H_{\alpha\beta}$ of any relative tensor of arbitrary weight κ, namely

$$\bar{H}_{\mu\nu}=(x\bar{x})^\kappa H_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu}, \tag{20.2}$$

where we assume that the determinent $|H_{\alpha\beta}|$ does not vanish. Forming the determinants of both members of these equations, we obtain

$$|\bar{H}_{\mu\nu}|=(x\bar{x})^{n\kappa+2}\,|H_{\alpha\beta}|. \tag{20.3}$$

When $n\kappa+2\neq 0$ we can use (20.3) to eliminate the Jacobian determinant $(x\bar{x})$ from (20.2), by which we have

$$\bar{J}_{\mu\nu}=J_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu}, \tag{20.4}$$

where
$$J_{\alpha\beta}=\frac{H_{\alpha\beta}}{|H|^r}, \quad r=\frac{\kappa}{n\kappa+2}.$$

From this point the methods for the formation of differential invariants may be applied as in the case of the metric space. If, however, the quantity $n\kappa+2$ vanishes, equations (20.4) are no longer valid; then (20.3) becomes
$$|\bar{H}_{\alpha\beta}|=|H_{\alpha\beta}|,$$
and equations (20.2) are essentially the same as the equations of transformation of the components of the fundamental conformal-affine tensor. It is interesting to observe that the exceptional case of the equations (20.2), i.e. the case for which these equations do not lead to the construction of tensor differential invariants as such invariants are constructed in metric space, is precisely the case corresponding to the equations (20.1).

21. Affine representation of conformal spaces

Our problem is now to construct conformal invariants of tensor character on the basis of the fundamental equations (20.1). For this purpose we seek a representation of conformal space analogous in as far as this is possible to the representation $A^{\star}_{n+1}$ of the projective theory. We begin by constructing certain important transformation equations of conformal space as an aid in the attainment of this goal.

Let us first observe that the contravariant form of the equations (20.1) is

$$\bar{G}^{\mu\nu}=(x\bar{x})^{2/n}\,G^{\alpha\beta}\frac{\partial\bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial\bar{x}^{\nu}}{\partial x^{\beta}}, \tag{21.1}$$

where the quantity $G^{\alpha\beta}$ is defined as the cofactor of $G_{\alpha\beta}$ in the determinant $|G_{\alpha\beta}|$; hence the relations
$$G_{\alpha\beta}\,G^{\alpha\gamma}=\delta^{\gamma}_{\beta},$$
are satisfied.

Now differentiate equations (20.1), obtaining

$$\begin{aligned}\frac{\partial\bar{G}_{\mu\nu}}{\partial\bar{x}^{\xi}}=&-\frac{2}{n}\bar{G}_{\mu\nu}\psi_{\xi}+(x\bar{x})^{-2/n}\\ &\times\left\{\frac{\partial G_{\alpha\beta}}{\partial x^{\gamma}}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}\frac{\partial x^{\gamma}}{\partial\bar{x}^{\xi}}+G_{\alpha\beta}\frac{\partial^2x^{\alpha}}{\partial\bar{x}^{\mu}\partial\bar{x}^{\xi}}\frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}+G_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial^2x^{\beta}}{\partial\bar{x}^{\nu}\partial\bar{x}^{\xi}}\right\},\end{aligned} \tag{21.2}$$

where we have put
$$\psi_{\xi}:=\frac{\partial\log(x\bar{x})}{\partial\bar{x}^{\xi}}.$$

By interchange of indices in (21.2) we next deduce

$$\begin{aligned}&\frac{1}{2}\left(\frac{\partial\bar{G}_{\mu\nu}}{\partial\bar{x}^{\xi}}+\frac{\partial\bar{G}_{\nu\xi}}{\partial\bar{x}^{\mu}}-\frac{\partial\bar{G}_{\xi\mu}}{\partial\bar{x}^{\nu}}\right)\\ &\quad=(x\bar{x})^{-2/n}\left\{G_{\alpha\beta}\frac{\partial^2x^{\alpha}}{\partial\bar{x}^{\mu}\partial\bar{x}^{\xi}}\frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}+\frac{1}{2}\left(\frac{\partial G_{\alpha\beta}}{\partial x^{\gamma}}+\frac{\partial G_{\beta\gamma}}{\partial x^{\alpha}}-\frac{\partial G_{\alpha\gamma}}{\partial x^{\beta}}\right)\frac{\partial x^{\alpha}}{\partial\bar{x}^{\mu}}\frac{\partial x^{\beta}}{\partial\bar{x}^{\nu}}\frac{\partial x^{\gamma}}{\partial\bar{x}^{\xi}}\right\}\\ &\qquad-\frac{1}{n}\{\bar{G}_{\mu\nu}\psi_{\xi}+\bar{G}_{\nu\xi}\psi_{\mu}-\bar{G}_{\mu\xi}\psi_{\nu}\}.\end{aligned} \tag{21.3}$$

Then from (21.1) and (21.3) we have

$$\text{(21.4)} \qquad \bar{K}^{\alpha}_{\mu\nu}\frac{\partial x^{\xi}}{\partial \bar{x}^{\alpha}} = \frac{\partial^2 x^{\xi}}{\partial \bar{x}^{\mu}\partial \bar{x}^{\nu}} + K^{\xi}_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}}\frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} - \frac{1}{n}\left\{\frac{\partial x^{\xi}}{\partial \bar{x}^{\mu}}\psi_{\nu} + \frac{\partial x^{\xi}}{\partial \bar{x}^{\nu}}\psi_{\mu} - \bar{G}_{\mu\nu}\bar{G}^{\sigma\tau}\frac{\partial x^{\xi}}{\partial \bar{x}^{\sigma}}\psi_{\tau}\right\},$$

where (2)

$$K^{\xi}_{\alpha\beta} = \tfrac{1}{2}G^{\xi\gamma}\left(\frac{\partial G_{\beta\gamma}}{\partial x^{\alpha}} + \frac{\partial G_{\gamma\alpha}}{\partial x^{\beta}} - \frac{\partial G_{\alpha\beta}}{\partial x^{\gamma}}\right),$$

i.e. the quantities $K^{\xi}_{\alpha\beta}$ are Christoffel symbols based on the components $G_{\alpha\beta}$. The functions $K^{\xi}_{\alpha\beta}$ are symmetric in their lower indices; also the identity

$$K^{\xi}_{\xi\beta} = 0$$

is readily verified, e.g. this follows from identities of the type (13.15) since $|G_{\alpha\beta}|$ is equal to unity.

Now construct the quantities $F^{\xi}_{\alpha\beta\gamma}$ from the functions $K^{\xi}_{\alpha\beta}$ in the same way as the components $B^{\xi}_{\alpha\beta\gamma}$ of the curvature tensor are constructed from the components of the affine connection, i.e. we have

$$F^{\xi}_{\alpha\beta\gamma} = \frac{\partial K^{\xi}_{\alpha\beta}}{\partial x^{\gamma}} - \frac{\partial K^{\xi}_{\alpha\gamma}}{\partial x^{\beta}} + K^{\xi}_{\nu\gamma}K^{\nu}_{\alpha\beta} - K^{\xi}_{\nu\beta}K^{\nu}_{\alpha\gamma},$$

and it is but a matter of calculation to show that the $F^{\xi}_{\alpha\beta\gamma}$ transform according to the equations

$$\bar{F}^{\lambda}_{\mu\nu\eta} = F^{\xi}_{\alpha\beta\gamma}\frac{\partial \bar{x}^{\lambda}}{\partial x^{\xi}}\frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}}\frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}}\frac{\partial x^{\gamma}}{\partial \bar{x}^{\eta}} + \delta^{\lambda}_{\eta}C_{\mu\nu} - \delta^{\lambda}_{\nu}C_{\mu\eta} + \bar{G}_{\mu\nu}C^{\lambda}_{\eta} - \bar{G}_{\mu\eta}C^{\lambda}_{\nu} + \frac{1}{n^2}(\delta^{\lambda}_{\eta}\bar{G}_{\mu\nu} - \delta^{\lambda}_{\nu}\bar{G}_{\mu\eta})\psi_{\sigma}\psi^{\sigma},$$

where

$$C_{\mu\nu} = -\frac{1}{n}\bar{K}^{\xi}_{\mu\nu}\psi_{\xi} - \frac{1}{n^2}\psi_{\mu}\psi_{\nu} + \frac{1}{n}\frac{\partial \psi_{\mu}}{\partial \bar{x}^{\nu}},$$

$$C^{\lambda}_{\eta} = \bar{G}^{\lambda\xi}C_{\eta\xi}, \quad \psi^{\sigma} = \bar{G}^{\sigma\tau}\psi_{\tau}.$$

From the quantities $F^{\xi}_{\alpha\beta\gamma}$ we may now form the quantities $F_{\alpha\beta}$ and F defined by

$$F_{\alpha\beta} = F^{\xi}_{\alpha\beta\xi}, \quad F = G^{\alpha\beta}F_{\alpha\beta};$$

these quantities have the equations of transformation

$$\bar{F}_{\mu\nu} = F_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}}\frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} + (n-2)C_{\mu\nu} + \bar{G}_{\mu\nu}C^{\xi}_{\xi} + \left(\frac{n-1}{n^2}\right)\bar{G}_{\mu\nu}\psi_{\sigma}\psi^{\sigma},$$

$$\bar{F} = (x\bar{x})^{2/n}F + 2(n-1)C^{\xi}_{\xi} + \left(\frac{n-1}{n}\right)\psi_{\sigma}\psi^{\sigma}.$$

We shall also need an expression

$$Q_{\alpha\beta} = F_{\alpha\beta} - \frac{F}{2(n-1)}G_{\alpha\beta},$$

having equations of transformation

$$(21.5)\qquad \bar{Q}_{\mu\nu}=Q_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu}-\left(\frac{n-2}{n}\right)\left\{\bar{K}^\xi_{\mu\nu}\psi_\xi+\frac{1}{n}\psi_\mu\psi_\nu-\frac{\partial\psi_\mu}{\partial\bar{x}^\nu}-\frac{1}{2n}\bar{G}_{\mu\nu}\psi_\sigma\psi^\sigma\right\}.$$

Since the quantities $F_{\alpha\beta}$ and $G_{\alpha\beta}$ are symmetric in the indices α, β, it follows that $Q_{\alpha\beta}$ are likewise. Finally we shall use the quantities

$$Q^\alpha_\beta = G^{\alpha\xi}Q_{\beta\xi},$$

which transform by the equations

$$(21.6)\qquad \bar{Q}^\mu_\nu=(x\bar{x})^{2/n}Q^\alpha_\beta\frac{\partial\bar{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial\bar{x}^\nu}$$

$$-\left(\frac{n-2}{n}\right)\left\{\bar{K}^\xi_{\beta\nu}\bar{G}^{\beta\mu}\psi_\xi+\frac{1}{n}\psi^\mu\psi_\nu-\bar{G}^{\xi\mu}\frac{\partial\psi_\xi}{\partial\bar{x}^\nu}-\frac{\delta^\mu_\nu}{2n}\psi_\sigma\psi^\sigma\right\}.$$

So far in this chapter all indices have had values $1, \ldots, n$. It will now be found helpful to agree that

Capital Greek letters denote indices $1, \ldots, n$;
Small Greek letters denote indices $0, 1, \ldots, n$;
Small Latin letters denote indices $0, 1, \ldots, n, \infty$.

We shall consider this convention to hold through the remainder of this chapter unless the contrary is stated, or it is otherwise evident that the rule cannot apply.*

Now consider the quantity u^i_k defined as the element in the ith row and kth column of the matrix

$$\left|\begin{array}{c|ccc|c}
1 & \psi_1 & \cdots & \psi_n & \frac{1}{2}\psi_\Sigma\psi^\Sigma \\
\hline
0 & \frac{\partial x^1}{\partial\bar{x}^1} & \cdots & \frac{\partial x^1}{\partial\bar{x}^n} & \frac{\partial x^1}{\partial\bar{x}^\Sigma}\psi^\Sigma \\
\vdots & \cdots & \cdots & \cdots & \vdots \\
0 & \frac{\partial x^n}{\partial\bar{x}^1} & \cdots & \frac{\partial x^n}{\partial\bar{x}^n} & \frac{\partial x^n}{\partial\bar{x}^\Sigma}\psi^\Sigma \\
\hline
0 & 0 & \cdots & 0 & (x\bar{x})^{2/n}
\end{array}\right|$$

To any transformation (1.2) of the group $\mathfrak{G}$ there corresponds a set of quantities u^i_k given as elements of the above matrix. If

$$T_1 : x^\Delta = f^\Delta(\bar{x}^1, \ldots, \bar{x}^n),$$
$$T_2 : \bar{x}^\Delta = \phi^\Delta(\bar{\bar{x}}^1, \ldots, \bar{\bar{x}}^n),$$
$$T_3 : x^\Delta = \omega^\Delta(\bar{\bar{x}}^1, \ldots, \bar{\bar{x}}^n),$$

* The symbol ∞ seems more convenient to use than the symbol $n+1$ employed by some writers.

i.e. if T_1 and T_2 are two transformations belonging to the group $\mathfrak{G}$ and T_3 is their resultant, then

$$u^i_k v^k_j = w_j,$$

where the v^k_j and w^i_j are determined from the transformations T_2 and T_3 in exactly the same manner as the u^i_k are determined by the transformation T_1. It follows in particular that

$$u^i_k \bar{u}^k_j = \delta^i_j, \quad u^i_k \bar{u}^j_i = \delta^j_k,$$

where the $\bar{u}^k_j$ are determined by the transformation inverse to (1.2). In other words the *quantities u^i_k combine in the same way as the derivatives determined by a coordinate transformation.* Now it is important to observe that the u^α_β are in fact deducible as the derivatives of a coordinate transformation, namely the transformation

$$x^0 = \bar{x}^0 + \log(x\bar{x}) + \text{const.}, \quad x^\Delta = f^\Delta(\bar{x}^1, \ldots, \bar{x}^n)$$

of the group ${}^*\mathfrak{G}$ occurring in the projective theory; this fact is intimately connected with our representation of the conformal space.

Let us now define a set of functions ${}^0\Gamma^i_{k\beta}$ by the equations(3)

$$\begin{aligned}
&{}^0\Gamma^i_{k0} = -\frac{1}{n}\delta^i_k, \quad {}^0\Gamma^i_{0\gamma} = -\frac{1}{n}\delta^i_\gamma,\\
&\left\{\begin{aligned}
&{}^0\Gamma^\Delta_{\Pi\Sigma} = K^\Delta_{\Pi\Sigma}, \quad {}^0\Gamma^0_{\Pi\Sigma} = \left(\frac{n}{n-2}\right)Q_{\Pi\Sigma}, \quad {}^0\Gamma^\infty_{\Pi\Sigma} = -\frac{1}{n}G_{\Pi\Sigma},\\
&{}^0\Gamma^\Delta_{\infty\Sigma} = \left(\frac{n}{n-2}\right)Q^\Delta_\Sigma, \quad {}^0\Gamma^0_{\infty\Sigma} = {}^0\Gamma^\infty_{\infty\Sigma} = 0.
\end{aligned}\right.
\end{aligned}$$

These functions ${}^0\Gamma^i_{k\beta}$ transform according to the equations

$$ {}^0\bar{\Gamma}^q_{k\sigma} u^i_q = \frac{\partial u^i_k}{\partial \bar{x}^\sigma} + {}^0\Gamma^i_{j\beta} u^j_k u^\beta_\sigma; \tag{21.7}$$

in fact these equations give the transformation equations

(20.1) for $i, k, \sigma = \infty, \Delta, \Sigma$ respectively,
(21.4) for $i, k, \sigma = \Delta, \Sigma, \Gamma$ respectively,
(21.5) for $i, k, \sigma = 0, \Delta, \Sigma$ respectively,
(21.6) for $i, k, \sigma = i, \infty, \Delta$ respectively.

The remaining equations (21.7) are satisfied identically.

Since the quantity $n-2$ appears in the denominators of several of the above expressions which define the functions ${}^0\Gamma^i_{k\sigma}$, the equations (21.7) will no longer be valid in case $n=2$; the assumption $n \geqq 3$ will therefore be made throughout the following developments of this chapter. See § 89 for a discussion of the case $n=2$.

The existence of a system of equations of the form (21.7) shows that we can represent the conformal space C_n as an $(n+1)$-dimensional space A^0_{n+1}

possessing an affine connection* with components ${}^0\Gamma^i_{k\beta}$, and whose coordinates $x^0, x^1, \ldots, x^n$ are subject to transformations of the group ${}^*\mathfrak{G}$. Since the indices i, k of the components ${}^0\Gamma^i_{k\beta}$ assume the range $0, 1, \ldots, n, \infty$, the affine connection of the space A^0_{n+1} must define the infinitesimal parallel displacements of a vector with $n+2$ components. A vector of the type in question will be known as a *conformal vector*; preliminary to writing down the explicit equations for the infinitesimal parallel displacement of such a vector we shall give the general definition of a relative conformal tensor in the following section.

22. Conformal tensors and invariants

Definition. *A set of functions $T^{i\ldots j}_{k\ldots l}(x)$ of the coordinates $x^1, \ldots, x^n$ of the region $\mathfrak{R}$ constitute the components of a relative conformal tensor T of weight W with respect to the x coordinate system of the affine representation A_{n+1} provided that the functions $T^{i\ldots j}_{k\ldots l}$ transform according to the equations*

$$\overline{T}^{i\ldots j}_{k\ldots l}(\overline{x})\,u^p_i \ldots u^q_j = |\,u^a_b\,|^W\, T^{p\ldots q}_{r\ldots s}(x)\,u^r_k \ldots u^s_l, \tag{22.1}$$

when the coordinates $x^1, \ldots, x^n$ undergo a transformation of the group $\mathfrak{G}$. The object obtained by abstraction from the above components with respect to the totality of coordinate systems of the affine representation A^0_{n+1}, whose coordinates are related by transformations of the group ${}^*\mathfrak{G}$, is called the tensor T. If, however, one or more of the indices in these equations are limited to the restricted range $0, 1, \ldots, n$, the entity T will be called an *incomplete conformal tensor of weight W.*

Infinitesimal parallel displacement of a vector of weight zero with components λ^i in the affine representation A^0_{n+1} produces a differential change in the components of this vector in accordance with the equations

$$d\lambda^i = -\,{}^0\Gamma^i_{k\beta}\lambda^k dx^\beta.$$

These equations are invariant in form under transformations of the coordinates of the representation A^0_{n+1}; in fact we have

$$d\lambda^i + {}^0\Gamma^i_{k\beta}\lambda^k dx^\beta = u^i_k[d\overline{\lambda}^k + {}^0\overline{\Gamma}^k_{m\sigma}\,\overline{\lambda}^m d\overline{x}^\sigma],$$

as can easily be deduced.

A conformal tensor differential invariant of order r and weight W is the entity whose components

$$\Gamma^{p\ldots q}_{r\ldots s}\left({}^0\Gamma^i_{k\beta};\ \frac{\partial\,{}^0\Gamma^i_{k\beta}}{\partial x^\gamma};\ \ldots,\ \frac{\partial^r\,{}^0\Gamma^i_{k\beta}}{\partial x^{\gamma_1}\ldots\partial x^{\gamma_r}}\right)$$

retain their form as functions of the ${}^0\Gamma$'s and their derivatives under transformations (22.1). We proceed now to the problem of constructing invariants of this character.

* It is to be observed that this is an affine connection in the general sense of § 2 in spite of the difference of range of indices appearing in the symbol of its components.

A simple addition to the components of the conformal-affine tensor of § 20 enables us to construct a conformal tensor invariant of order zero. If we define the functions $\mathfrak{C}_{ik}$ as the elements of the matrix

$$\left|\begin{array}{c|ccc|c} 0 & 0 & \cdots & 0 & -1 \\ \hline 0 & G_{11} & \cdots & G_{1n} & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 0 & G_{n1} & \cdots & G_{nn} & 0 \\ \hline -1 & 0 & \cdots & 0 & 0 \end{array}\right|$$

and then observe that

$$|u_b^a| = (x\bar{x})^{\frac{n+2}{n}},$$

we see that

$$\bar{\mathfrak{C}}_{kl} = |u_b^a|^{\frac{-2}{n+2}} \mathfrak{C}_{rs} u_k^r u_l^s. \tag{22.2}$$

Hence the quantities $\mathfrak{C}_{ik}$ constitute the components of a symmetric conformal tensor invariant of weight $-2/(n+2)$ and order zero; we call this invariant the *fundamental conformal tensor* $\mathfrak{C}$.

If we differentiate the equations (21.7) with respect to $\bar{x}^\tau$ and then proceed to form the equations of integrability after the manner of deducing the equations of transformation of the components of the curvature tensor in § 12, we obtain

$$^0\bar{B}^q_{i\mu\nu} u^i_q = {}^0B^i_{k\alpha\beta} u^k_j u^\alpha_\mu u^\beta_\nu, \tag{22.3}$$

where

$$^0B^i_{k\alpha\beta} = \frac{\partial\, ^0\Gamma^i_{k\alpha}}{\partial x^\beta} - \frac{\partial\, ^0\Gamma^i_{k\beta}}{\partial x^\alpha} + {}^0\Gamma^i_{j\beta}\, {}^0\Gamma^j_{k\alpha} - {}^0\Gamma^i_{j\alpha}\, {}^0\Gamma^j_{k\beta}. \tag{22.4}$$

Equations (22.3) show the existence of an incomplete conformal tensor, with components $^0B^i_{k\alpha\beta}$, which we shall call the *incomplete conformal curvature tensor*.

We may observe that the components $^0B^i_{k\alpha\beta}$ satisfy the following identities

$$^0B^i_{i\alpha\beta} = {}^0B^i_{k\alpha 0} = {}^0B^i_{k0\beta} = {}^0B^i_{0\alpha\beta} = {}^0B^\infty_{k\alpha\beta} = {}^0B^0_{\infty\alpha\beta} = 0. \tag{22.5}$$

In view of these identities the equations (22.3) yield

$$\bar{Y}^\Sigma_{\Theta\Psi\Pi} \frac{\partial x^\Omega}{\partial \bar{x}^\Sigma} = Y^\Omega_{\Lambda\Delta\Phi} \frac{\partial x^\Lambda}{\partial \bar{x}^\Theta} \frac{\partial x^\Delta}{\partial \bar{x}^\Psi} \frac{\partial x^\Phi}{\partial \bar{x}^\Pi},$$

where (4)

$$Y^\Omega_{\Lambda\Delta\Phi} = {}^0B^\Omega_{\Lambda\Delta\Phi}.$$

Thus the functions $Y^\Omega_{\Lambda\Delta\Phi}$ constitute the components of an affine tensor which we will call the *conformal-affine curvature tensor*. Expansion of $Y^\Omega_{\Lambda\Delta\Phi}$ gives

$$\frac{1}{n-2}\Big\{(n-2) F^\Omega_{\Lambda\Delta\Phi} + \delta^\Omega_\Delta F_{\Lambda\Phi} - \delta^\Omega_\Phi F_{\Lambda\Delta} + G^{\Omega\Sigma}(F_{\Sigma\Delta} G_{\Lambda\Phi} - F_{\Sigma\Phi} G_{\Lambda\Delta}) + \frac{F}{n-1}(\delta^\Omega_\Phi G_{\Lambda\Delta} - \delta^\Omega_\Delta G_{\Lambda\Phi})\Big\}.$$

By substituting into this expression the value of the F's in terms of the $K^{\Phi}_{\Lambda\Delta}$ and their derivatives, and then eliminating the K's by means of the substitution

$$K^{\Phi}_{\Lambda\Delta}=\Gamma^{\Phi}_{\Lambda\Delta}-\frac{1}{n}(\delta^{\Phi}_{\Lambda}\Gamma^{\Sigma}_{\Sigma\Delta}+\delta^{\Phi}_{\Delta}\Gamma^{\Sigma}_{\Sigma\Lambda}-g_{\Lambda\Delta}g^{\Phi\Omega}\Gamma^{\Sigma}_{\Sigma\Omega}),$$

we obtain after some reductions

$$Y^{\Omega}_{\Lambda\Delta\Phi}=B^{\Omega}_{\Lambda\Delta\Phi}+\frac{1}{n-2}(\delta^{\Omega}_{\Delta}B_{\Lambda\Phi}-\delta^{\Omega}_{\Phi}B_{\Lambda\Delta})+\frac{1}{n-2}(B^{\Omega}_{\Delta}\,g_{\Lambda\Phi}-B^{\Omega}_{\Phi}\,g_{\Lambda\Delta})$$
$$-\frac{B}{(n-1)(n-2)}(\delta^{\Omega}_{\Delta}\,g_{\Lambda\Phi}-\delta^{\Omega}_{\Phi}\,g_{\Lambda\Delta}),$$

where the $B^{\Omega}_{\Lambda\Delta\Phi}$ are the components of the affine curvature tensor formed from the Christoffel symbols $\Gamma^{\Phi}_{\Lambda\Delta}$, and the $B_{\Lambda\Delta}$ and B are the contracted curvature tensors defined in § 52. From the above expression we easily obtain the identity

(22.6) $$Y^{\Omega}_{\Lambda\Delta\Omega}=0.$$

By a similar expansion of (22.4) we obtain

(22.7) $${}^{0}B^{0}_{\Lambda\Delta\Phi}=\left(\frac{n}{n-2}\right)\left[\frac{\partial F_{\Lambda\Delta}}{\partial x^{\Phi}}-\frac{\partial F_{\Lambda\Phi}}{\partial x^{\Delta}}+K^{\Omega}_{\Lambda\Delta}F_{\Omega\Phi}-K^{\Omega}_{\Lambda\Phi}F_{\Omega\Delta}\right.$$
$$\left.-\frac{1}{2(n-1)}\left(G_{\Lambda\Delta}\frac{\partial F}{\partial x^{\Phi}}-G_{\Lambda\Phi}\frac{\partial F}{\partial x^{\Delta}}\right)\right]$$
$$=-\Gamma^{\Sigma}_{\Sigma\Psi}\,Y^{\Psi}_{\Lambda\Delta\Phi}+\left(\frac{n}{n-2}\right)(B_{\Lambda\Delta,\,\Phi}-B_{\Lambda\Phi,\,\Delta})$$
$$-\frac{n}{2(n-1)(n-2)}(g_{\Lambda\Delta}B_{,\,\Phi}-g_{\Lambda\Phi}B_{,\,\Delta})$$
$$=\left(\frac{n}{n-3}\right)Y^{\Sigma}_{\Lambda\Delta\Phi,\,\Sigma}-\Gamma^{\Sigma}_{\Sigma\Psi}\,Y^{\Psi}_{\Lambda\Delta\Phi},$$

and also $${}^{0}B^{\Sigma}_{\Omega\Delta\Phi}=G^{\Sigma\Lambda}\,{}^{0}B^{0}_{\Lambda\Delta\Phi}.$$

When $n=3$ the conformal-affine curvature tensor vanishes;* from the equations (22.7) we then obtain

$$\bar{Z}_{\Theta\Lambda\Pi}=Z_{\Delta\Sigma\Omega}\frac{\partial x^{\Delta}}{\partial\bar{x}^{\Theta}}\frac{\partial x^{\Sigma}}{\partial\bar{x}^{\Lambda}}\frac{\partial x^{\Omega}}{\partial\bar{x}^{\Pi}},$$

where $$Z_{\Delta\Sigma\Omega}={}^{0}B^{0}_{\Delta\Sigma\Omega}.$$

The affine tensor having the components $Z_{\Delta\Sigma\Omega}$ will be called the *conformal-affine curvature tensor for the three dimensional conformal space* C_3.

Making use of equations analogous to (13.1(b)), we obtain

$$\frac{\partial\,|\,u^{a}_{b}\,|^{W}}{\partial\bar{x}^{\alpha}}=W\,|\,u^{a}_{b}\,|^{W}\{{}^{0}\bar{\Gamma}^{h}_{h\alpha}-{}^{0}\Gamma^{h}_{h\sigma}u^{\sigma}_{\alpha}\};$$

we then find by differentiating (22.1) with respect to $\bar{x}^{\alpha}$ and eliminating derivatives of the u^{i}_{k} which arise, by means of (21.7), that

(22.8) $$\bar{T}^{i\dots j}_{k\dots l,\alpha}u^{p}_{i}\dots u^{q}_{j}=|\,u^{a}_{b}\,|^{W}T^{p\dots q}_{r\dots s,\mu}u^{r}_{k}\dots u^{s}_{l}u^{\mu}_{\alpha},$$

* This fact can be established by expanding the above expression for the components $Y^{\Omega}_{\Lambda\Delta\Phi}$ corresponding to what was done for the case of the projective-affine curvature tensor ($n=2$); see p. 58; this calculation can, however, be simplified by choosing special coordinates, cf. L. P. Eisenhart, *Riemannian Geometry*, p. 91.

where

$$T^{p\dots q}_{r\dots s,\mu}=\frac{\partial T^{p\dots q}_{r\dots s}}{\partial x^{\mu}}+T^{h\dots q}_{r\dots s}\,{}^{0}\Gamma^{p}_{h\mu}+\dots+T^{p\dots h}_{r\dots s}\,{}^{0}\Gamma^{q}_{h\mu}$$
$$-T^{p\dots q}_{h\dots s}\,{}^{0}\Gamma^{h}_{r\mu}-\dots-T^{p\dots q}_{r\dots h}\,{}^{0}\Gamma^{h}_{s\mu}-W\,T^{p\dots q}_{r\dots s}\,{}^{0}\Gamma^{h}_{h\mu}. \tag{22.9}$$

The quantities $T^{p\dots q}_{r\dots s,\mu}$ are therefore the components of an incomplete conformal tensor which will be called the *incomplete covariant derivative of the tensor* T. On account of the restricted range $0, 1, \dots, n$ of the index μ appearing in the components $T^{p\dots q}_{r\dots s,\mu}$ of the incomplete covariant derivative, the above process of covariant differentiation cannot be applied to the equations (22.8) to form a second covariant derivative of tensor character. This suggests that we "complete" the above covariant derivative by defining the values of its components for the value ∞ of the index added by the process of covariant differentiation; we consider this question in the following section.

23. Completion of the incomplete covariant derivative. General case

We can form a conformal tensor from the incomplete conformal tensor whose components occur in (22.8) if we can define a set of quantities $T^{p\dots q}_{r\dots s,\infty}$, depending on the coordinates $x^1, \dots, x^n$ alone, and having equations of transformation

$$\bar{T}^{i\dots}_{k\dots l,\infty}u^{p}_{i}\dots u^{q}_{j}$$
$$=|u^{a}_{b}|^{W}\left\{T^{p\dots q}_{r\dots s,\infty}u^{\infty}_{\infty}+T^{p\dots q}_{r\dots s,0}u^{0}_{\infty}+\sum_{\Delta=1}^{n}T^{p\dots q}_{r\dots s,\Delta}u^{\Delta}_{\infty}\right\}u^{r}_{k}\dots u^{s}_{l}. \tag{23.1}$$

In fact (22.8) and (23.1) combine to give

$$\bar{T}^{i\dots j}_{k\dots l,f}u^{p}_{i}\dots u^{q}_{j}=|u^{a}_{b}|^{W}T^{p\dots q}_{r\dots s,g}u^{r}_{k}\dots u^{s}_{l}u^{g}_{f}; \tag{23.2}$$

the tensor whose components appear in these equations will be called the complete covariant derivative, or simply the *covariant derivative of the tensor* T [5].

Let N denote the number of indices $p, \dots, q$ and M the number of indices $r, \dots, s$ appearing in the components of the above tensor T. Then the equations (22.9) show that

$$T^{p\dots q}_{r\dots s,0}=\left[\frac{M-N+(n+2)\,W}{n}\right]T^{p\dots q}_{r\dots s}. \tag{23.3}$$

To construct the desired quantities $T^{p\dots q}_{r\dots s,\infty}$ transforming by (23.1), we now differentiate (22.8) with respect to $\bar{x}^{\beta}$, obtaining

$$\bar{T}^{i\dots j}_{k\dots l,\alpha\beta}u^{p}_{i}\dots u^{q}_{j}$$
$$=|u^{a}_{b}|^{W}\{T^{p\dots q}_{r\dots s,\mu\nu}u^{\mu}_{\alpha}u^{\nu}_{\beta}+T^{p\dots q}_{r\dots s,\sigma}\,{}^{0}\Gamma^{\infty}_{\alpha\beta}u^{\sigma}_{\infty}\}u^{r}_{k}\dots u^{s}_{l}, \tag{23.4}$$

where
$$T^{p\dots q}_{r\dots s,\mu\nu}=\frac{\partial T^{p\dots q}_{r\dots s,\mu}}{\partial x^\nu}+T^{h\dots q}_{r\dots s,\mu}\,{}^0\Gamma^p_{h\nu}+\dots+T^{p\dots h}_{r\dots s,\mu}\,{}^0\Gamma^q_{h\nu}$$
$$-T^{p\dots q}_{h\dots s,\mu}\,{}^0\Gamma^h_{r\nu}-\dots-T^{p\dots q}_{r\dots h,\mu}\,{}^0\Gamma^h_{s\nu}$$
$$-T^{p\dots q}_{r\dots s,\sigma}\,{}^0\Gamma^\sigma_{\mu\nu}-WT^{p\dots q}_{r\dots s,\mu}\,{}^0\Gamma^h_{h\nu}.$$
In particular we can deduce from these latter equations and (23.3) that

(23.5) $$T^{p\dots q}_{r\dots s,0\nu}=\left[\frac{M-N+(n+2)W+1}{n}\right]T^{p\dots q}_{r\dots s,\nu},$$

(23.6) $$T^{p\dots q}_{r\dots s,\mu 0}=\left[\frac{M-N+(n+2)W+1}{n}\right]T^{p\dots q}_{r\dots s,\mu},$$

(23.7) $$T^{p\dots q}_{r\dots s,00}=\left[\frac{M-N+(n+2)W+1}{n}\right]\left[\frac{M-N+(n+2)W}{n}\right]T^{p\dots q}_{r\dots s}.$$

Now restrict the values α and β in (23.4) to the range $1,\dots,n$ and rewrite the right members of these equations so as to obtain a corresponding restriction of range for the indices μ and ν, i.e. we form the equations

(23.8) $$\overline{T}^{i\dots j}_{k\dots l,\Delta\Sigma}\overset{p}{u_i}\dots\overset{q}{u_j}=|\overset{a}{u_b}|^W\{T^{p\dots q}_{r\dots s,\Lambda\Pi}\overset{\Lambda}{u_\Delta}\overset{\Pi}{u_\Sigma}+T^{p\dots q}_{r\dots s,0\Pi}\overset{0}{u_\Delta}\overset{\Pi}{u_\Sigma}$$
$$+T^{p\dots q}_{r\dots s,\Lambda 0}\overset{\Lambda}{u_\Delta}\overset{0}{u_\Sigma}+T^{p\dots q}_{r\dots s,00}\overset{0}{u_\Delta}\overset{0}{u_\Sigma}$$
$$+T^{p\dots q}_{r\dots s,\Pi}\,{}^0\overline{\Gamma}^\infty_{\Delta\Sigma}\overset{\Pi}{u_\infty}+T^{p\dots q}_{r\dots s,0}\,{}^0\overline{\Gamma}^\infty_{\Delta\Sigma}\overset{0}{u_\infty}\}\overset{r}{u_k}\dots\overset{s}{u_l}.$$

If we multiply (23.8) by $\overline{G}^{\Delta\Sigma}$ and sum on the indices Δ and Σ, taking account of (21.1), (23.3), (23.5), (23.6), (23.7), we obtain a system of equations which can be written

(23.9) $$\overline{T}^{i\dots j}_{k\dots l,\Delta\Sigma}\overline{G}^{\Delta\Sigma}\overset{p}{u_i}\dots\overset{q}{u_j}$$
$$=|\overset{a}{u_b}|^W\Big\{T^{p\dots q}_{r\dots s,\Lambda\Pi}G^{\Lambda\Pi}\overset{\infty}{u_\infty}+\left[\frac{2M-2N+2(n+2)W+2-n}{n}\right]$$
$$\times(T^{p\dots q}_{r\dots s,0}\overset{0}{u_\infty}+T^{p\dots q}_{r\dots s,\Pi}\overset{\Pi}{u_\infty})\Big\}\overset{r}{u_k}\dots\overset{s}{u_l}.$$

Hence if the constant K defined by
$$K=2M-2N+2(n+2)W+2-n$$
does not vanish, we can put

(23.10) $$T^{p\dots q}_{r\dots s,\infty}=\left(\frac{n}{K}\right)\sum_{\Lambda=1}^{n}\sum_{\Pi=1}^{n}T^{p\dots q}_{r\dots s,\Lambda\Pi}G^{\Lambda\Pi},$$

and the components $T^{p\dots q}_{r\dots s,\infty}$ so defined will transform by (23.1). We thus obtain the complete covariant derivative with components $T^{p\dots q}_{r\dots s,g}$ transforming in accordance with (23.2).

If $K=0$, equations (23.9) show the existence of a conformal tensor $\mathfrak{T}$ of weight $W+\frac{2}{n+2}$ having the components
$$\sum_{\Lambda=1}^{n}\sum_{\Pi=1}^{n}T^{p\dots q}_{r\dots s,\Lambda\Pi}G^{\Lambda\Pi}.$$
The constant K will not vanish for the tensor $\mathfrak{T}$, so that it is possible to construct the complete conformal covariant derivative of this tensor. It is, in fact, evident that we can form the infinite sequence of successive covariant derivatives of the tensor $\mathfrak{T}$.

24. An extension of the preceding method

In order to construct an infinite sequence of conformal invariants of tensor character, analogous to the sequence of covariant derivatives (13.18) of the affine curvature tensor, we must complete the incomplete conformal curvature tensor defined in § 22 by an extension of the preceding method. Let us in fact consider the general problem of completing a tensor D having components $D^{p\dots q}_{r\dots s\mu\nu}$ which are skew-symmetric in the indices μ and ν and such that

$$D^{p\dots q}_{r\dots s0\nu} \equiv D^{p\dots q}_{r\dots s\mu 0} \equiv 0.$$

Cf. equations (22.5). Differentiation of the equations of transformation of the components of the tensor D, namely

$$\overline{D}^{i\dots j}_{k\dots l\alpha\beta} u^p_i \dots u^q_j = |u^a_b|^W D^{p\dots q}_{r\dots s\mu\nu} u^r_k \dots u^s_l u^\mu_\alpha u^\nu_\beta, \tag{24.1}$$

and elimination of the derivatives of the u^i_k which occur, by (21.7), give

$$\begin{aligned}\overline{D}^{i\dots j}_{k\dots l\alpha\beta\gamma} u^p_i \dots u^q_j = |u^a_b|^W \{ & D^{p\dots q}_{r\dots s\mu\nu\pi} u^\mu_\alpha u^\nu_\beta u^\pi_\gamma \\ & + D^{p\dots q}_{r\dots s\mu\nu} ({}^0\overline{\Gamma}^\infty_{\alpha\gamma} u^\mu_\infty u^\nu_\beta + {}^0\overline{\Gamma}^\infty_{\beta\gamma} u^\mu_\alpha u^\nu_\infty)\} u^r_k \dots u^s_l,\end{aligned} \tag{24.2}$$

where
$$\begin{aligned} D^{p\dots q}_{r\dots s\mu\nu\pi} = \frac{\partial D^{p\dots q}_{r\dots s\mu\nu}}{\partial x^\pi} & + D^{h\dots q}_{r\dots s\mu\nu}\,{}^0\Gamma^p_{h\pi} + \dots + D^{p\dots h}_{r\dots s\mu\nu}\,{}^0\Gamma^q_{h\pi} \\ & - D^{p\dots q}_{h\dots s\mu\nu}\,{}^0\Gamma^h_{r\pi} - \dots - D^{p\dots q}_{r\dots h\mu\nu}\,{}^0\Gamma^h_{s\pi} \\ & - D^{p\dots q}_{r\dots s\sigma\nu}\,{}^0\Gamma^\sigma_{\mu\pi} - D^{p\dots q}_{r\dots s\mu\sigma}\,{}^0\Gamma^\sigma_{\nu\pi} - W D^{p\dots q}_{r\dots s\mu\nu}\,{}^0\Gamma^h_{h\pi}.\end{aligned}$$

If as before N and M denote the number of indices $p, \dots, q$ and $r, \dots, s$, respectively, then it follows from the last formula that

$$D^{p\dots q}_{r\dots s0\nu\pi} = \frac{1}{n} D^{p\dots q}_{r\dots s\pi\nu},$$

$$D^{p\dots q}_{r\dots s\mu 0\pi} = \frac{1}{n} D^{p\dots q}_{r\dots s\mu\pi},$$

$$D^{p\dots q}_{r\dots s\mu\nu 0} = \left[\frac{M - N + (n+2)W + 2}{n}\right] D^{p\dots q}_{r\dots s\mu\nu}.$$

Equations (24.2) can accordingly be written

$$\begin{aligned}\overline{D}^{i\dots j}_{k\dots l\Delta\Theta\Lambda} u^p_i \dots u^q_j = |u^a_b|^W \Big\{ & D^{p\dots q}_{r\dots s\Pi\Sigma\Omega} \overset{\Pi}{u}_\Delta \overset{\Sigma}{u}_\Theta \overset{\Omega}{u}_\Lambda + \frac{1}{n} D^{p\dots q}_{r\dots s\Omega\Sigma} \overset{0}{u}_\Delta \overset{\Sigma}{u}_\Theta \overset{\Omega}{u}_\Lambda \\ & + \frac{1}{n} D^{p\dots q}_{r\dots s\Pi\Omega} \overset{\Pi}{u}_\Delta \overset{0}{u}_\Theta \overset{\Omega}{u}_\Lambda + \left[\frac{M - N + (n+2)W + 2}{n}\right] D^{p\dots q}_{r\dots s\Pi\Sigma} \overset{\Pi}{u}_\Delta \overset{\Sigma}{u}_\Theta \overset{0}{u}_\Lambda \\ & + D^{p\dots q}_{r\dots s\Pi\Sigma} ({}^0\overline{\Gamma}^\infty_{\Delta\Lambda} \overset{\Pi}{u}_\infty \overset{\Sigma}{u}_\Theta + {}^0\overline{\Gamma}^\infty_{\Theta\Lambda} \overset{\Pi}{u}_\Delta \overset{\Sigma}{u}_\infty) \Big\} u^r_k \dots u^s_l.\end{aligned} \tag{24.3}$$

Multiplying (24.3) by $\bar{G}^{\Theta\Lambda}$ and summing on the indices Θ and Λ, we obtain after some reductions

$$(24.4a)\qquad \bar{D}^{i\ldots j}_{k\ldots l\Delta\Theta\Lambda}\bar{G}^{\Theta\Lambda}u^p_i\ldots u^q_j$$

$$= |u^a_b|^W \left\{D^{p\ldots q}_{r\ldots s\Pi\Sigma\Omega}G^{\Sigma\Omega}u^\Pi_\Delta u^\infty_\infty + D^{p\ldots q}_{r\ldots s\Pi\Sigma}u^\Pi_\Delta u^\Sigma_\infty\right\}u^r_k\ldots u^s_l,$$

where the integer L is defined by

$$L = M - N + (n+2)W + 4 - n.$$

Since

$$D^{p\ldots q}_{r\ldots s0\Sigma\Omega}G^{\Sigma\Omega} = 0,$$

owing to the skew-symmetric property of the components D, it follows that (24.4a) holds also when the index Δ is replaced by α, i.e.

$$(24.4b)\qquad \bar{D}^{i\ldots j}_{k\ldots l\alpha\Theta\Lambda}\bar{G}^{\Theta\Lambda}u^p_i\ldots u^q_j$$

$$= |u^a_b|^W \left\{D^{p\ldots q}_{r\ldots s\mu\Sigma\Omega}G^{\Sigma\Omega}u^\mu_\alpha u^\alpha_\alpha + \left(\frac{L}{n}\right)D^{p\ldots q}_{r\ldots s\mu\Sigma}u^\mu_\alpha u^\Sigma_\infty\right\}u_k\ldots u_l.$$

If $L \neq 0$, we put

$$D^{p\ldots q}_{r\ldots s\mu\infty} \equiv -D^{p\ldots q}_{r\ldots s\infty\mu} \equiv \left(\frac{n}{L}\right)D^{p\ldots q}_{r\ldots s\mu\Sigma\Omega}G^{\Sigma\Omega},$$

$$D^{p\ldots q}_{r\ldots s0\infty} \equiv D^{p\ldots q}_{r\ldots s\infty 0} \equiv D^{p\ldots q}_{r\ldots s\infty\infty} = 0.$$

Then (24.1) and (24.4b) combine to give

$$\bar{D}^{i\ldots j}_{k\ldots lcd}u^p_i\ldots u^q_j = |u^a_b|^W D^{p\ldots q}_{r\ldots sfg}u^r_k\ldots u^s_l u^f_c u^g_d,$$

and we have succeeded in completing the incomplete conformal tensor D; from their definition the components of the complete conformal tensor D are skew-symmetric in their last two indices, i.e.

$$D^{p\ldots q}_{r\ldots sfg} \equiv -D^{p\ldots q}_{r\ldots sgf}.$$

1°. When $L = 0$, equations (24.4b) lead to the definition of an incomplete conformal tensor of weight $W + \dfrac{2}{n+2}$; it is obvious that a method analogous to the above can in general be used to complete this tensor after which its successive covariant derivatives can be constructed.

2°. As a first application of the method of this section we can consider the components of an incomplete conformal tensor D to be defined by

$$(24.5)\qquad D^{p\ldots q}_{r\ldots s\mu\nu} = T^{p\ldots q}_{r\ldots s,\mu\nu} - T^{p\ldots q}_{r\ldots s,\nu\mu}.$$

Cf. equation (23.4). The tensor D is then the direct analogue of the skew-symmetric part of the second covariant derivative of a given affine tensor. If $K = 0$ so that the above method of completing the incomplete covariant derivative of the tensor T does not apply, we can still complete the tensor D whose components are defined by (24.5) provided that the constant L does not vanish for this latter tensor; in fact only if $n = 6$ can the constants K and L be equal to zero simultaneously.

25. Systems algebraically equivalent to the system of equations of transformation of the components of a conformal tensor

From a strictly analytical standpoint it is clear that the process of covariant differentiation, whether applied to an affine, projective or conformal tensor, is essentially a process for the construction of conditions of integrability, in tensorial form, of the equations of transformation of the components of the given tensor. This suggests that we replace these latter equations, in the exceptional case $K=0$ or $L=0$ when the method of § 23 or § 24 does not apply, by an algebraically equivalent system of tensor character before performing the operation of covariant differentiation. We shall consider in this section several types of equivalent systems of this sort with a view to applying them to the problem in the above exceptional cases.

Denote by T an arbitrarily given tensor of weight W having the components $T^{p\dots q}_{r\dots s}$, and consider the conformal tensor with components

$$T^{p\dots q\,c\dots d}_{r\dots s\,e\dots f}=T^{p\dots q}_{r\dots s}T^{c\dots d}_{e\dots f}; \tag{25.1}$$

this latter tensor is called the *square of the tensor* T and will be denoted by the symbol T^2. In the equations of transformation of the components of T^2, namely

$$\overline{T}^{i\dots j\,g\dots h}_{k\dots l\,v\dots w}u^p_i\dots u^d_h=|\,u^a_b\,|^{2W}\,T^{p\dots q\,c\dots d}_{r\dots s\,e\dots f}u^r_k\dots u^f_w, \tag{25.2}$$

let us take the indices $k,\dots,l$ to be identical with the indices $v,\dots,w$, and the indices $p,\dots,q$ identical with the indices $c,\dots,d$. Then (25.2) reduces to

$$\overline{T}^{i\dots j}_{k\dots l}u^p_i\dots u^q_j=\pm[|\,u^a_b\,|^W T^{p\dots q}_{r\dots s}u^r_k\dots u^s_l],$$

and these equations are identical, when the plus sign is selected in the right members, with the equations of transformation of the components of the tensor T. Equations (25.2) therefore furnish an example of a system algebraically equivalent to the equations of transformation of the components of the tensor T.

If we define the tensor T^m as the mth power of the tensor T by equations analogous to (25.1), then it is evident that the equations of transformation of the components of T^m are likewise algebraically equivalent to the equations of transformation of the components of the tensor T; in fact, for m odd, even the above ambiguity of algebraic sign does not appear. It is evident therefore that the equations of transformation of the components of T^m can be used in place of the equations of transformation of the components of T in the process of forming the conditions of integrability of these latter equations, i.e., the process of covariant differentiation.

Another type of algebraically equivalent system can be obtained by observing that the constants V_i defined by

$$V_\alpha\equiv 0,\quad V_\infty\equiv 1$$

constitute the components of a conformal covariant vector of weight $-2/(n+2)$, i.e. these quantities undergo the transformation

$$\overline{V}_k = | u^a_b |^{-\frac{2}{n+2}} V_i u^i_k .$$

Now define

(25.3) $$T^{p \dots q}_{r \dots st} = V_t T^{p \dots q}_{r \dots s} ,$$

so that

(25.4) $$\overline{T}^{i \dots j}_{k \dots lf} u^p_i \quad u^q_j = | u^a_b |^{W - \frac{2}{n+2}} T^{p \dots q}_{r \dots st} u^r_k \dots u^s_l u^t_f .$$

If $f \neq \infty$ in (25.4) both members of these equations vanish identically, while if $f = \infty$ these equations become identical with the equations of transformation of the components of the tensor T. Hence (25.4) can be used in place of the equations of transformation of the components of the tensor T in the process of constructing the conditions of integrability of these latter equations.

We will refer to the tensor with components given by (25.3) as the *augmented tensor* T. The vector with components V_i will be called the *relative numerical vector* V. We can also define an *absolute numerical vector* V by the equations

(25.5) $$V^i = \mathfrak{G}^{ik} V_k ,$$

where the quantities $\mathfrak{G}^{ik}$, given as the elements of the matrix

$$\begin{array}{c|c|ccc|c|}
0 & 0 & 0 & \dots & 0 & -1 \\
\hline
1 & 0 & G^{11} & \dots & G^{1n} & 0 \\
\vdots & \vdots & \dots & \dots & \dots & \vdots \\
n & 0 & G^{n1} & \dots & G^{nn} & 0 \\
\hline
\infty & -1 & 0 & \dots & 0 & 0
\end{array}$$

are the components of the contravariant form of the fundamental conformal tensor defined in § 22; this tensor is of weight $2/(n+2)$. Then $V^\infty \equiv V^\Delta \equiv 0$, also $V^0 \equiv -1$, and these components satisfy the relations

$$V^i = \overline{V}^k u^i_k .$$

We see that these latter equations will hold for any value of the component V^0, but if this component is taken to be different from -1 the relations (25.5) will not be satisfied. It is obvious that a tensor might be defined analogous to the above augmented tensor T, by using the absolute numerical vector V in place of the relative numerical vector V.

26. Exceptional case $K=0$

The process defined in § 23 for constructing the complete covariant derivative of the tensor T with components $T^{p\dots q}_{r\dots s}$ is applicable provided that the number K, given by

$$K = 2M - 2N + 2(n+2)W + 2 - n,$$

is different from zero; if, now, $K=0$ for the above tensor T, then $K=-2$ for the augmented tensor T. Hence the complete covariant derivative of the augmented tensor T can be formed.

Using the formulae (22.9) and (23.3), it follows readily that

$$T^{p\dots q}_{r\dots s\infty,\,\Delta} \equiv T^{p\dots q}_{r\dots s,\,\Delta}, \tag{26.1}$$

$$T^{p\dots q}_{r\dots s\infty,\,0} \equiv \left[\frac{M-N+(n+2)W-1}{n}\right] T^{p\dots q}_{r\dots s}, \tag{26.2}$$

$$T^{p\dots q}_{r\dots s\Delta,\,\Sigma} \equiv \frac{1}{n} T^{p\dots q}_{r\dots s} G_{\Delta\Sigma}, \tag{26.3}$$

$$T^{p\dots q}_{r\dots s0,\,\Sigma} \equiv T^{p\dots q}_{r\dots s\mu,\,0} \equiv 0, \tag{26.4}$$

where the left members of these equations represent components of the covariant derivative of the augmented tensor T. Now consider certain of the equations of transformation of the components of this latter tensor, namely

$$\overline{T}^{i\dots j}_{k\dots l,\infty\alpha} u^p_i \dots u^q_j = |u^a_b|^{W-\frac{2}{n+2}} T^{p\dots q}_{r\dots st,\,v} u^r_k \dots u^s_l u^t_\infty u^v_\alpha. \tag{26.5}$$

If $\alpha=0$ and $n=4$, we find when use is made of (26.2) and the fact that $K=0$ for the tensor T, that both members of the equations (26.5) vanish identically; otherwise these equations are exactly equivalent to the equations of transformation of the components of the incomplete covariant derivative of the tensor T. On this account, as well as on account of the equations (26.1) and (26.2), the process of forming the covariant derivative of the augmented tensor T is closely analogous to the process of completing the incomplete covariant derivative of the tensor T as defined in § 23, and for that reason can be looked upon as a new method of completing this latter tensor. We will therefore refer to the above covariant derivative of the augmented tensor T as the *complete covariant derivative* $K=0$, *or simply as the covariant derivative,* $K=0$, *of the tensor* T. Since $K=0$ likewise for this latter tensor, the sequence of the components of the successive covariant derivatives of the tensor T has the form

$$T^{p\dots q}_{r\dots s,\,w_1w_2};\quad T^{p\dots q}_{r\dots s,\,w_1\dots w_4};\quad T^{p\dots q}_{r\dots s,\,w_1\dots w_6};\quad \dots.$$

The number of covariant indices in the symbols of these components increases by jumps of 2 and the weights of the corresponding tensors by amounts of $-2/(n+2)$ as we proceed from left to right along the sequence.

It follows from what was said in § 25 as well as from what was just said regarding equations (26.5) that the equations of transformation of the components in the above sequence express the conditions of integrability obtained by successive differentiation of the equations of transformation of the components of the original tensor T.

If $K=0$ for the tensor T, then the number K is equal to $(n-2)(m-1)$ for the tensor T^m defined in the preceding paragraph. Hence, for $m \geqq 2$, the number $K>0$ for the tensor T^m and we can construct the complete covariant derivative of this tensor; since, moreover, the effect of the covariant differentiation is to increase the number K by 2, it is seen that K will never equal zero in the process of forming the successive covariant derivatives of T^m.

27. Exceptional case $L=0$

If $L=0$ for the incomplete conformal tensor D defined in § 24, then $L=-1$ for the *incomplete augmented tensor* D defined by the components

$$D^{p \dots q}_{r \dots st\mu\nu} = V_t D^{p \dots q}_{r \dots s\mu\nu}.$$

This latter tensor can therefore be completed by the method of § 24 and the value of the number K is equal to $n-4$ for the *complete augmented tensor* D so obtained. Hence if $n=4$ we must construct covariant derivatives, $K=0$, of the augmented tensor D; otherwise the process of covariant differentiation of § 23 applies.

28. The complete conformal curvature tensor and its successive covariant derivatives

The method of § 24 can be used in general to complete the incomplete curvature tensor defined in § 22. That is, we put

$$D^i_{k\alpha\beta} = {}^0B^i_{k\alpha\beta};$$

the requisite conditions on the components $D^i_{k\alpha\beta}$ are then satisfied since the components ${}^0B^i_{k\alpha\beta}$ in addition to being skew-symmetric in the indices α and β are equal to zero by (22.5) whenever the values $\alpha=0$ or $\beta=0$ are assumed. For this case $L=4-n$ and hence we can use the method of § 24 to complete the curvature tensor whenever n is different from 4; we thus obtain the components ${}^0B^i_{klm}$ of the complete conformal curvature tensor $(n \neq 4)$.

In case $n=4$ we apply the method of § 27 to obtain the complete conformal curvature tensor $(n=4)$ with the components ${}^0B^i_{jklm}$; this tensor is of weight $-2/(n+2)$. Since the constant $K=0$ for the conformal curvature tensor $(n=4)$, we must form its successive covariant derivatives by the method of § 26.

Now consider the complete conformal curvature tensor $(n \neq 4)$; for this tensor we have $N=1$ and $W=0$ so that the value $2M-n$ is assumed by the number K. Hence if n is an odd integer, greater than or equal to 3, we can construct the infinite sequence of complete covariant derivatives of the

complete conformal curvature tensor by the method of § 23. Let us indicate this by writing

$$\boxed{n \text{ odd}} \quad {}^0B^i_{klm};\ {}^0B^i_{klmp};\ {}^0B^i_{klmpq},\ \ldots$$

as the components of the curvature tensor and its successive covariant derivatives.

If n is even and greater than 4, the constant K will always vanish at some stage of the process of forming the infinite sequence of complete covariant derivatives of the curvature tensor. For example, if $n=6$, $K=0$ for the conformal curvature tensor; if $n=8$, we can construct the first complete conformal covariant derivative of the conformal curvature tensor by the method of § 23, but then find that $K=0$ for this covariant derivative, etc. The following indications for $n=6$, 8 and 10 enable us to see at a glance the behaviour of these sequences for the above and higher values of the dimensionality number.

$$\boxed{n=6} \quad {}^0B^i_{klm};$$

$$\boxed{n=8} \quad {}^0B^i_{klm};\ {}^0B^i_{klmp};$$

$$\boxed{n=10} \quad {}^0B^i_{klm};\ {}^0B^i_{klmp};\ {}^0B^i_{klmp}$$

To continue these sequences we must apply the process of covariant differentiation of § 26 which, as we have already remarked, must likewise be applied to form the covariant derivatives of the complete conformal curvature tensor ($n=4$). For all even values of the dimensionality number n the continued sequences are therefore represented by the following scheme

$$\boxed{n=4} \quad {}^0B^i_{jklm};\ {}^0B^i_{jklmw_1w_2};\ {}^0B^i_{jklmw_1\ldots w_4};$$

$$\boxed{n=6} \quad {}^0B^i_{jkl};\ {}^0B^i_{jklw_1w_2};\ {}^0B^i_{jklw_1\ldots w_4};$$

$$\boxed{n=8} \quad {}^0B^i_{jkl};\ {}^0B^i_{jklm};\ {}^0B^i_{jklmw_1w_2};$$

Each of these sequences begins with the complete conformal curvature tensor; and those tensors whose components appear along the main diagonal of the above infinite matrix of components are each of weight $-2/(n+2)$.

REFERENCES

(1) T. Y. Thomas, "Invariants of relative quadratic differential forms", *Proc. N.A.S.* **11** (1925), pp. 722–5.

(2) The functions $K^\alpha_{\beta\gamma}$ were originally found in a different form by J. M. Thomas as the conformal analogue of the components $\Pi^\alpha_{\beta\gamma}$ of the projective connection. He also defined the components $F^\epsilon_{\alpha\beta\gamma}$ and determined their equations of transformation. See "Conformal correspondence of Riemann spaces", *Proc. N.A.S.* **11** (1925), pp. 257–9.

(3) The components ${}^0\Gamma^i_{k\sigma}$ were defined and used to determine the incomplete conformal curvature tensor by T. Y. Thomas, "On conformal geometry", *Proc. N.A.S.* **12** (1926), pp. 352–9.

(4) The conformal-affine curvature tensor was first obtained by H. Weyl, who showed that this tensor vanishes for $n=3$; more precisely the components of Weyl's

tensor are obtainable from the components of the conformal-affine curvature tensor defined in § 22 by multiplying these latter components by the constant factor $2-n$. See H. Weyl, ref. (2), Chapter I, p. 404. The components of the conformal-affine curvature tensor for the three dimensional conformal space C_3 result by multiplying by $n/(n-2)$ a set of quantities obtained by É. Cotton who showed that the vanishing of these quantities is a necessary and sufficient condition that a three dimensional Riemann space can be mapped conformally on a Euclidean space. See "Sur les variétés à trois dimensions", *Ann. Fac. des Sciences Toulouse* (2), **1** (1899), p. 410. Cf. J. A. Schouten, *Der Ricci-Kalkül*, p. 170; also J. M. Thomas, "Conformal invariants", *Proc. N.A.S.* **12** (1926), p. 393.

(5) The method of completing a tensor which appears in §§ 23–8 was given by T. Y. Thomas, "Conformal tensors", Notes I and II, *Proc. N.A.S.* **18** (1932), pp. 103–12, 189–93. O. Veblen completed the covariant derivative by imposing a set of invariant conditions involving components of conformal tensors. These conditions correspond to our equations (23.10) but differ from these latter equations in the important respect that they involve only derivatives of the first order. See "Differential invariants and geometry", *Atti del congresso internazionale dei matematici* (1), **6** (1928), pp. 181–9; also "Conformal tensors and connections", *Proc. N.A.S.* **14** (1928), pp. 735–45. Under certain conditions this method enables one to complete the covariant derivative of a tensor T in a very satisfactory manner. However, Veblen's method does not lead to a sequence of conformal tensor invariants analogous to the Riemann curvature tensor and its successive covariant derivatives by means of which the equivalence or non-equivalence of two conformal spaces can be determined as in Chapter VIII.

CHAPTER V

NORMAL COORDINATES

29. Affine normal coordinates

We shall call by the generic name of *normal coordinates* any system of coordinates which are in general defined only throughout the immediate neighbourhood of a point P of the region $\mathscr{R}$ covered by the arbitrary x coordinates, but which are distinguishable from these latter coordinates by certain special properties. An example of such coordinates is already to be found in the affine normal coordinates y^α defined by equations (3.2), which possess the property that the equations of a path of the affine space passing through the origin have the same form as the equations of a straight line in ordinary Euclidean space with reference to rectangular Cartesian coordinates (1).

The affine normal coordinates y^α are determined uniquely by the x coordinate system and a point P, this latter point being identified with the origin of the normal coordinate system. Let us now transform the x^α coordinates by a transformation (1.2) belonging to the group $\mathfrak{G}$ and let us then define the normal coordinates $\bar{y}^\alpha$ which are determined by the $\bar{x}$ coordinate system and the above point P. We seek the analytical relation between the coordinates y^α and $\bar{y}^\alpha$ of these two normal coordinate systems. By the transformation (1.2), the equations $x^\alpha = \phi^\alpha(s)$ of a path C determined by the initial conditions

$$x^\alpha = p^\alpha, \quad \frac{dx^\alpha}{ds} = \xi^\alpha, \quad \text{for} \quad s = 0,$$

go over into the equations $\bar{x}^\alpha = \psi^\alpha(s)$, and the above initial conditions become

$$\bar{x}^\alpha = \bar{p}^\alpha, \quad \frac{d\bar{x}^\alpha}{ds} = \bar{\xi}^\alpha, \quad \text{for} \quad s = 0;$$

here

$$p^\alpha = f^\alpha(\bar{p}^1, \ldots, \bar{p}^n),$$

and

(29.1) $$\xi^\alpha = a^\alpha_\beta \bar{\xi}^\beta,$$

where

(29.2) $$a^\alpha_\beta \qquad \left(\partial \bar{x}^\beta\right)_P,$$

i.e. the a's are the values of the derivatives of the coordinate transformation (1.2) evaluated at the point P. Now $y^\alpha = \xi^\alpha s$ and $\bar{y}^\alpha = \bar{\xi}^\alpha s$ are respectively the equations of the path C with reference to the y and $\bar{y}$ normal coordinate

systems. Hence if we multiply equations (29.1) through by the parameter s, we obtain

$$y^{\alpha}=a^{\alpha}_{\beta}\bar{y}^{\beta} \tag{29.3}$$

along the path C. But any point Q in a sufficiently small neighbourhood of P is joined to P by a unique path; hence the relation (29.3) holds throughout the neighbourhood of the point P and we have the following result.

When the coordinates x^{α} undergo a transformation of the group $\mathfrak{G}$, the affine normal coordinates determined by the x coordinate system and a point P suffer a linear homogeneous transformation (29.3) *with constant coefficients.*

In other words the normal coordinates y^{α} are transformed like the components of a contravariant vector. They do not, however, define a vector in the narrow sense, but are the components of a "step" from the origin of the normal coordinates to the point at which the coordinates are taken. An arbitrary step determined by the points P and Q can be represented by the coordinates of the point Q in the normal coordinate system associated with the point P.

There is an alternative method of treating affine normal coordinates which is of some interest. Let us suppose that the components $\Gamma^{\alpha}_{\beta\gamma}(x)$ become $C^{\alpha}_{\beta\gamma}(y)$ in the normal coordinate system, so that

$$C^{\sigma}_{\beta\gamma}\frac{\partial x^{\alpha}}{\partial y^{\sigma}}=\frac{\partial^2 x^{\alpha}}{\partial y^{\beta}\partial y^{\gamma}}+\Gamma^{\alpha}_{\mu\nu}\frac{\partial x^{\mu}}{\partial y^{\beta}}\frac{\partial x^{\nu}}{\partial y^{\gamma}}; \tag{29.4}$$

the equations of the paths in normal coordinates will then be given by

$$\frac{d^2 y^{\alpha}}{ds^2}+C^{\alpha}_{\beta\gamma}\frac{dy^{\beta}}{ds}\frac{dy^{\gamma}}{ds}=0. \tag{29.5}$$

Since the equations of a path through the origin are of the form $y^{\alpha}=\xi^{\alpha}s$, we obtain from (29.5) that

$$C^{\alpha}_{\beta\gamma}\xi^{\beta}\xi^{\gamma}=0$$

along this path. Multiplying these latter equations through by s^2 shows that

$$C^{\alpha}_{\beta\gamma}y^{\beta}y^{\gamma}=0, \tag{29.6}$$

and these equations must hold throughout the normal coordinate system since they are true for all paths through the origin; in other words *equations* (29.6) *are satisfied identically in the coordinates y^{α}.*

Equations (29.6) can be used to define the normal coordinates y^{α}. We have from (29.4) and (29.6) that

$$\left(\frac{\partial^2 x^{\alpha}}{\partial y^{\beta}\partial y^{\gamma}}+\Gamma^{\alpha}_{\mu\nu}\frac{\partial x^{\mu}}{\partial y^{\beta}}\frac{\partial x^{\nu}}{\partial y^{\gamma}}\right)y^{\beta}y^{\gamma}=0. \tag{29.7}$$

These differential equations uniquely determine a functional relation

between the x^α and the y^α when taken in conjunction with the initial conditions

$$y^\alpha = 0 \quad \text{when} \quad x^\alpha = p^\alpha,$$

$$\frac{\partial x^\alpha}{\partial y^\beta} = \delta^\alpha_\beta \quad \text{when} \quad x^\alpha = p^\alpha.$$

In fact when we differentiate (29.7) repeatedly and substitute these initial conditions, we obtain the equations (3.2) in which the coefficients Γ have the values previously deduced in § 3.

To prove the relation (29.3), between the normal coordinates y^α and $\bar{y}^\alpha$, as a result of their method of definition by means of the differential equations (29.7), we first construct the equations of transformation between the components $C^\alpha_{\beta\gamma}$ and $\bar{C}^\alpha_{\beta\gamma}$ of the affine connection in these two normal coordinate systems, namely

$$\bar{C}^\sigma_{\beta\gamma} \frac{\partial y^\alpha}{\partial \bar{y}^\sigma} = \frac{\partial^2 y^\alpha}{\partial \bar{y}^\beta \partial \bar{y}^\gamma} + C^\alpha_{\mu\nu} \frac{\partial y^\mu}{\partial \bar{y}^\beta} \frac{\partial y^\nu}{\partial \bar{y}^\gamma}. \tag{29.8}$$

Multiplying each side of (29.8) by $\bar{y}^\beta \bar{y}^\gamma$ and summing on the indices β and γ, we obtain

$$\left(\frac{\partial^2 y^\alpha}{\partial \bar{y}^\beta \partial \bar{y}^\gamma} + C^\alpha_{\mu\nu} \frac{\partial y^\mu}{\partial \bar{y}^\beta} \frac{\partial y^\nu}{\partial \bar{y}^\gamma}\right) \bar{y}^\beta \bar{y}^\gamma = 0 \tag{29.9}$$

in consequence of the equations (29.6) with reference to the $\bar{y}$ normal coordinate system. By the definition of the normal coordinates, the relation between the y^α and $\bar{y}^\alpha$ must be such as to satisfy the conditions

$$y^\alpha = 0, \quad \frac{\partial y^\alpha}{\partial \bar{y}^\beta} = a^\alpha_\beta, \quad \text{when} \quad \bar{y}^\alpha = 0, \tag{29.10}$$

where the constants a^α_β are defined by (29.2). The fact that (29.3) constitutes the relation between the normal coordinates y^α and $\bar{y}^\alpha$ then follows by observing first that (29.3) satisfies (29.9) and the conditions (29.10), and second that (29.9) has a unique solution satisfying the conditions (29.10).

The equations (29.6) *characterize the coordinates y^α as affine normal coordinates, i.e. if $C^\alpha_{\beta\gamma}$ are the components of affine connection in a system of coordinates y^α and if the equations* (29.6) *are satisfied identically, the y^α are affine normal coordinates.* To see this let us determine the normal coordinate system $\tilde{y}$ having its origin at the origin of the y coordinate system. The normal coordinates $\tilde{y}^\alpha$ will then be given as solutions of the system of differential equations

$$\left(\frac{\partial^2 y^\alpha}{\partial \tilde{y}^\beta \partial \tilde{y}^\gamma} + C^\alpha_{\mu\nu} \frac{\partial y^\mu}{\partial \tilde{y}^\beta} \frac{\partial y^\nu}{\partial \tilde{y}^\gamma}\right) \tilde{y}^\beta \tilde{y}^\gamma = 0,$$

subject to the initial conditions

$$y^\alpha = 0, \quad \frac{\partial y^\alpha}{\partial \tilde{y}^\beta} = \delta^\alpha_\beta, \quad \text{when} \quad \tilde{y}^\alpha = 0.$$

Hence $y^\alpha = \bar{y}^\alpha$ since this satisfies the initial conditions and also the differential equations on account of (29.6); it follows that the y^α are affine normal coördinates.

Other equations besides (29.6) can be found which will characterize the normal coordinates y^α in the particular case of a metric space; this circumstance arises from the fact that the components $\Gamma^\alpha_{\beta\gamma}$ are then the Christoffel symbols based on the quantities $g_{\alpha\beta}$. Hence equations (29.6) reduce immediately to the equations

$$\left(2\frac{\partial \psi_{\alpha\beta}}{\partial y^\gamma} - \frac{\partial \psi_{\beta\gamma}}{\partial y^\alpha}\right) y^\beta y^\gamma = 0, \tag{29.11}$$

where the $\psi_{\alpha\beta}$ denote the components of the fundamental metric tensor in the normal coordinates y^α. Now the relation (5.13), holding along any path of the space, implies the relation

$$\psi_{\alpha\beta}\xi^\alpha\xi^\beta = (\psi_{\alpha\beta})_0\xi^\alpha\xi^\beta \tag{29.12}$$

along a path C defined by $y^\alpha = \xi^\alpha s$ through the origin of the normal coordinate system; hence

$$\psi_{\alpha\beta}y^\alpha y^\beta = (\psi_{\alpha\beta})_0 y^\alpha y^\beta \tag{29.13}$$

in the system of normal coordinates y^α. Differentiating (29.13) we obtain

$$\frac{\partial \psi_{\beta\gamma}}{\partial y^\alpha} y^\beta y^\gamma + 2\psi_{\alpha\beta}y^\beta - 2(\psi_{\alpha\beta})_0 y^\beta = 0,$$

and these equations when combined with (29.11) yield

$$\frac{\partial \psi_{\alpha\beta}}{\partial y^\gamma} y^\beta y^\gamma + \psi_{\alpha\beta}y^\beta - (\psi_{\alpha\beta})_0 y^\beta = 0. \tag{29.14}$$

Now along the above path C we observe that

$$\frac{d}{ds}[\psi_{\alpha\beta}y^\beta - (\psi_{\alpha\beta})_0 y^\beta] = 0$$

in consequence of (29.14). Hence

$$\psi_{\alpha\beta}y^\beta = (\psi_{\alpha\beta})_0 y^\beta, \tag{29.15}$$

since C is an arbitrary path through the origin of the normal coordinate system. *The equations* (29.15) *characterize the coordinates y^α as affine normal coordinates.* To prove this we have merely to differentiate (29.15) with respect to y^γ and then multiply the resulting equations by y^γ and y^α in turn. We thus find the two following sets of equations

$$\frac{\partial \psi_{\alpha\beta}}{\partial y^\gamma} y^\beta y^\gamma = 0, \tag{29.16}$$

$$\frac{\partial \psi_{\alpha\beta}}{\partial y^\gamma} \qquad = 0. \tag{29.17}$$

It can be shown that the *coordinates y^α are likewise characterized as affine normal coordinates by the equations* (29.16). Observe that along the path C we have

$$\frac{d}{ds}[\psi_{\alpha\beta}\xi^\beta - (\psi_{\alpha\beta})_0\xi^\beta] = \frac{\partial \psi_{\alpha\beta}}{\partial y^\gamma}\xi^\beta\xi^\gamma = 0$$

on account of (29.16). By the above argument it therefore follows that the equations (29.15) are satisfied. We shall have occasion to use this result in § 41.

30. Absolute normal coordinates

The existence of the configurations consisting of the n fundamental vectors with components h^α_i at each point P of the region $\mathcal{R}$ of the affine space of distant parallelism suggests that we attempt to define a normal coordinate system with origin at an arbitrary point P of $\mathcal{R}$ and with coordinate axes

along the directions determined by the fundamental vectors at P. This requirement in itself does not lead to a unique determination of a coordinate system and we shall therefore combine it with certain other requirements as stated in the following postulates: the coordinate system so defined will be called an *absolute normal coordinate system* for a reason which will appear later(2).

POSTULATES OF THE ABSOLUTE NORMAL COORDINATE SYSTEM

A. *With each point P of the region $\mathcal{R}$ of the space of distant parallelism there is associated a normal coordinate system z having its origin at the point P.*

B. *The coordinate axes of the normal system z at the point P are tangent to the directions determined by the fundamental vectors at P, the coordinates z^i being so chosen that the conditions*

$$\frac{\partial x^\alpha}{\partial z^i} = h^\alpha_i \tag{30.1}$$

are satisfied at P.

C. *The paths (6.11) which pass through the origin of the normal system z have the form*

$$z^i = \xi^i s, \tag{30.2}$$

where the ξ^i are constants.

The above postulates give a complete geometrical characterization of the coordinates z^i of the absolute normal coordinate system. If we denote the components of affine connection $\Lambda^\alpha_{\beta\gamma}$ by $\lambda^i_{jk}(z)$ when referred to the absolute normal coordinate system, we have in consequence of Postulate C that

$$\lambda^i_{jk}\xi^j\xi^k = 0$$

along a path through the origin of the absolute normal system. It follows immediately from these latter equations that the equations

$$\lambda^i_{jk}z^j z^k = 0 \tag{30.3}$$

are satisfied identically in the absolute normal system. When we replace the components λ^i_{jk} in (30.3) by their values in terms of the components $\Lambda^\alpha_{\beta\gamma}$ we obtain a system of partial differential equations for the determination of the coordinates x^α in terms of the coordinates z^i, namely,

$$\left(\frac{\partial^2 x^\alpha}{\partial z^j \partial z^k} + \Lambda^\alpha_{\beta\gamma}\frac{\partial x^\beta}{\partial z^j}\frac{\partial x^\gamma}{\partial z^k}\right) z^j z^k = 0. \tag{30.4}$$

This system of equations possesses a unique solution $x^\alpha = \phi^\alpha(z)$ satisfying a set of initial conditions

$$x^\alpha = p^\alpha \quad \text{when} \quad z^i = 0, \tag{30.5}$$

$$\frac{\partial x^\alpha}{\partial z^i} = p^\alpha_i \quad \text{when} \quad z^i = 0, \tag{30.6}$$

where p^α and p^α_i are arbitrary constants.

It is obvious that the system of differential equations (30.4) possesses a unique formal solution

$$x^\alpha = p^\alpha + p_i^\alpha z^i + \frac{1}{2!} p_{ij}^\alpha z^i z^j + \ldots, \tag{30.7}$$

such that the conditions (30.5) and (30.6) are satisfied. A proof of the convergence of the series (30.7) can be given in the following manner. Consider a system of equations

$$\frac{\partial^2 X^\alpha}{\partial z^j \partial z^k} - F_{\beta\gamma}^\alpha \frac{\partial X^\beta}{\partial z^j} \frac{\partial X^\gamma}{\partial z^k} = 0, \tag{30.8}$$

where the $F_{\beta\gamma}^\alpha$ are analytic functions of the variables X^α in the neighbourhood of the values $X^\alpha = P^\alpha$. The conditions of integrability of (30.8) are that

$$\frac{\partial F_{\beta\gamma}^\alpha}{\partial X^\delta} - \frac{\partial F_{\beta\delta}^\alpha}{\partial X^\gamma} + F_{\mu\delta}^\alpha F_{\beta\gamma}^\mu - F_{\mu\gamma}^\alpha F_{\beta\delta}^\mu = 0$$

identically. These conditions are satisfied by taking all the functions $F_{\beta\gamma}^\alpha$ equal to one another in accordance with the equations

$$F_{\beta\gamma}^\alpha = \frac{M}{1 - \dfrac{(X^1 - P^1) + \ldots + (X^n - P^n)}{\rho}},$$

where M and ρ are constants; the system (30.8) then possesses a unique solution

$$X^\alpha = P^\alpha + P_i^\alpha z^i + \frac{1}{2!} P_{ij}^\alpha z^i z^j + \ldots, \tag{30.9}$$

satisfying the conditions

$$X^\alpha = P^\alpha \quad \text{when } z^i = 0,$$

$$\frac{\partial X^\alpha}{\partial z^i} = P_i^\alpha \quad \text{when } z^i = 0.$$

Now choose the above constants M and ρ so that each function $F_{\beta\gamma}^\alpha$ dominates the corresponding function $\Lambda_{\beta\gamma}^\alpha$ and, furthermore, choose the P^α and P_i^α such that

$$P^\alpha \geqq |p^\alpha|, \quad P_i^\alpha \geqq |p_i^\alpha|.$$

In view of the fact that (30.9) is likewise a solution of the system

$$\left(\frac{\partial^2 X^\alpha}{\partial z^j \partial z^k} - F_{\beta\gamma}^\alpha \frac{\partial X^\beta}{\partial z^j} \frac{\partial X^\gamma}{\partial z^k}\right) z^j z^k = 0,$$

it is then easily seen that each expansion (30.9) dominates the corresponding expansion (30.7). The convergence of the expansions (30.7), within a sufficiently small neighbourhood of the values $z^j = 0$, is therefore established.

By Postulate A condition (30.5) is satisfied, provided that the constants p^α denote the coordinates of the point P; moreover the values $h_i^\alpha(p)$ are to be ascribed to the above constants p_i^α in consequence of (30.1). By repeated differentiation of (30.4) and use of the initial conditions (30.1) and (30.5) we determine the successive coefficients $H(p)$ of the power series expansions

$$x^\alpha = p^\alpha + h_i^\alpha(p) z^i - \frac{1}{2!} H_{ij}^\alpha(p) z^i z^j - \frac{1}{3!} H_{ijk}^\alpha(p) z^i z^j z^k - \ldots. \tag{30.10}$$

Thus we have that

$$H_{ij}^\alpha = \Lambda_{\beta\gamma}^\alpha h_i^\beta h_j^\gamma,$$

where

$$\Lambda_{\beta\gamma}^\alpha = \tfrac{1}{2} h_i^\alpha \left(\frac{\partial h_\beta^i}{\partial x^\gamma} + \frac{\partial h_\gamma^i}{\partial x^\beta}\right), \tag{30.11}$$

and
$$H^{\alpha}_{ijk}=\Lambda^{\alpha}_{\beta\gamma\delta}h^{\beta}_{i}h^{\gamma}_{j}h^{\delta}_{k},$$
where

$$\Lambda^{\alpha}_{\beta\gamma\delta}=\frac{1}{3}\left[\left(\frac{\partial\Lambda^{\alpha}_{\beta\gamma}}{\partial x^{\delta}}+\frac{\partial\Lambda^{\alpha}_{\gamma\delta}}{\partial x^{\beta}}+\frac{\partial\Lambda^{\alpha}_{\delta\beta}}{\partial x^{\gamma}}\right)-2\left(\Lambda^{\alpha}_{\sigma\beta}\Lambda^{\sigma}_{\gamma\delta}+\Lambda^{\alpha}_{\sigma\gamma}\Lambda^{\sigma}_{\delta\beta}+\Lambda^{\alpha}_{\sigma\delta}\Lambda^{\sigma}_{\beta\gamma}\right)\right], \tag{30.12}$$

etc. The Jacobian of (30.10) does not vanish since it is equal to the determinant $|h^{\alpha}_{i}(p)|$ at the origin of the absolute normal system; hence (30.10) possesses a unique inverse. Either the transformation (30.10), or its inverse, gives a unique definition of the absolute normal coordinate system.

Let us denote by $\bar{z}^{i}$ the coordinates of the absolute normal system determined by the point P and the components of affine connection $\bar{\Lambda}^{\alpha}_{\beta\gamma}(\bar{x})$ which result from the components $\Lambda^{\alpha}_{\beta\gamma}(x)$ by a transformation of the x^{α} coordinates belonging to the group $\mathfrak{G}$. The relation between the z^{i} and $\bar{z}^{i}$ coordinates must then be such that

$$z^{i}=0,\quad \frac{\partial z^{i}}{\partial\bar{z}^{k}}=\delta^{i}_{k},\quad \text{when}\quad \bar{z}^{i}=0. \tag{30.13}$$

Also this relation must be such as to satisfy the system of equations

$$\left(\frac{\partial^{2}z^{i}}{\partial\bar{z}^{j}\,\partial\bar{z}^{k}}+\lambda^{i}_{pq}\frac{\partial z^{p}}{\partial\bar{z}^{j}}\frac{\partial z^{q}}{\partial\bar{z}^{k}}\right)\bar{z}^{j}\bar{z}^{k}=0. \tag{30.14}$$

Hence

$$z^{i}=\bar{z}^{i}. \tag{30.15}$$

This follows from the fact that (30.15) satisfies (30.14) and the initial conditions (30.13), and that (30.14) possesses a unique solution which satisfies the conditions (30.13). We can therefore say *the coordinates z^{i} remain unchanged when the underlying coordinates x^{α} undergo an arbitrary transformation of the group* $\mathfrak{G}$. It is on account of this property that the normal coordinates z^{i} are called *absolute normal coordinates.*

In a similar manner we can show that *when the components of the fundamental vectors undergo a transformation* (6.5), *the absolute normal coordinates z^{i} associated with any point P likewise undergo a linear homogeneous transformation*

$$z^{i}=a^{i}_{k}z^{k}_{*}, \tag{30.16}$$

where the a^{i}_{k} are constants such that the determinant $|a^{i}_{k}|$ is not equal to zero.

Denoting by $A^{i}_{|j|}$ the covariant components of the fundamental vectors in the absolute normal coordinate system (see § 35), it follows immediately from Postulate B that

$$A^{i}_{|j|}=\delta^{i}_{j} \tag{30.17}$$

at the origin of this system. In case we are dealing with a metric space of distant parallelism, the fundamental quadratic differential form is then given by

$$ds^{2}=\sum_{1}^{n}e_{i}dz^{i}dz^{i}$$

on account of (30.17). Also (30.16) yields

$$\sum_{i=1}^{n} e_i z^i z^i = \sum_{i=1}^{n} e_i z^i_\star z^i_\star$$

in consequence of the conditions (6.8); hence the equations

$$\sum_{i=1}^{n} e_i z^i z^i = 0$$

are invariant under orthogonal transformations of the fundamental vectors in a metric space of distant parallelism.

By an argument analogous to that given in § 29 in which use is made of the initial conditions (30.17) we can show that the equations (30.3) characterize the coordinates z^i as absolute normal coordinates. We observe that the equations

$$\frac{\partial A^i_{|j|}}{\partial z^k} z^j z^k = 0 \tag{30.18}$$

are completely equivalent to (30.3). Hence *the equations* (30.18) *characterize the coordinates z^i as absolute normal coordinates*; use will be made of this fact in § 42.

31. Projective normal coordinates

By analogy with the equations (29.6) we are led to attempt to characterize a system of normal coordinates in the projective space of paths P_n, by a system of equations of the form

$$\Xi^i_{jk} \eta^j \eta^k = 0, \tag{31.1}$$

where Ξ^i_{jk} denotes the components of projective connection in the η coordinate system. Expansion of these equations, corresponding to (29.7), gives

$$\left[\frac{\partial^2 x^i}{\partial \eta^j \partial \eta^k} + \Pi^i_{ab} \frac{\partial x^a}{\partial \eta^j} \frac{\partial x^b}{\partial \eta^k} - \frac{1}{n+1}\left(\frac{\partial x^i}{\partial \eta^j} \frac{\partial \log (x\eta)}{\partial \eta^k} + \frac{\partial x^i}{\partial \eta^k} \frac{\partial \log (x\eta)}{\partial \eta^j}\right)\right] \eta^j \eta^k = 0 \tag{31.2}$$

as the equations to determine the relation between the x and η coordinates.

Now it is at once evident that repeated differentiation of (31.2), followed by evaluation at $\eta^i = 0$, does not lead directly to the determination of the successive coefficients of the power series expansions of the x^i in terms of the η^i variables, corresponding to the case of the equations (29.7); in fact the determination of these coefficients by this method involves a complicated formal procedure. We shall overcome this difficulty in the following manner. Let us put

$$\frac{\partial x^0}{\partial \eta^i} = \frac{\partial \log (x\eta)}{\partial \eta^i} \quad (i = 1, \ldots, n); \tag{31.3}$$

then equations (31.2) can be written

$$\left(\frac{\partial^2 x^i}{\partial \eta^j \partial \eta^k} + {}^\star\Gamma^i_{\mu\nu} \frac{\partial x^\mu}{\partial \eta^j} \frac{\partial x^\nu}{\partial \eta^k}\right) \eta^j \eta^k = 0, \tag{31.4}$$

where the ${}^*\Gamma^{\alpha}_{\beta\gamma}$ are as defined in § 16. Here we have adopted the convention which shall be used throughout the remainder of this section, that Latin indices have the range $1, \ldots, n$ and Greek indices the range $0, 1, \ldots, n$. Having obtained the equations (31.4) we shall no longer consider that a derivative $\partial x^0/\partial\eta^i$ in them is equal to the right member of (31.3), but shall regard this derivative as the derivative of a new quantity x^0 which we wish to determine as a function of the variables η^i. It then becomes necessary to add a new equation to the set (31.4), which we shall assume to have the form

$$\left\{\frac{\partial^2 x^0}{\partial\eta^j\,\partial\eta^k}+[jk]\right\}\eta^j\eta^k=0, \tag{31.5}$$

where the quantity in the brackets can depend on the ${}^*\Gamma^{\sigma}_{\mu\nu}$, the first derivatives $\partial x^\sigma/\partial\eta^i$ and the second derivatives which appear in (31.4). The set of equations consisting of (31.4) and (31.5) will now have the desired property: namely, repeated differentiation of these equations and subsequent evaluation at $\eta^i=0$ will determine directly the coefficients of the required power series expansions.

Equations (31.4) and (31.5) will have a solution of the form

$$x^\alpha=\omega^\alpha(\eta^1, \ldots, \eta^n), \tag{31.6}$$

determined uniquely by the initial conditions

$$x^\alpha=p^\alpha, \quad \frac{\partial x^\alpha}{\partial\eta^i}=\delta^\alpha_i, \quad \text{when} \quad \eta^i=0. \tag{31.7}$$

We wish to choose the bracket expression in (31.5) so that this solution (31.6) will be such that (31.3) is satisfied, where the expression $(x\eta)$ stands for the Jacobian determinant formed from the n functions ω^i. Then

$$x^i=\omega^i(\eta^1, \ldots, \eta^n) \tag{31.8}$$

will constitute a solution of the original system (31.2) such that

$$x^i=p^i, \quad \frac{\partial x^i}{\partial\eta^j}=\delta^i_j, \quad \text{when} \quad \eta^i=0. \tag{31.9}$$

Let us now consider the equations (31.4) and (31.5) as equations involving the coordinates x^α of the $(n+1)$-dimensional affine representation $A^\star_{n+1}$ of the projective space P_n and let us make the transformation

$$x^i=\omega^i(\eta^1, \ldots, \eta^n), \quad x^0=\eta^0+\omega^0(\eta^1, \ldots, \eta^n) \tag{31.10}$$

to a new system of coordinates η^α; the transformation equations inverse to (31.10) are

$$\eta^i=\Omega^i(x^1, \ldots, x^n), \quad \eta^0=x^0+\Omega^0(x^1, \ldots, x^n). \tag{31.11}$$

Our problem is then to choose the bracket expression in (31.5) so that the transformation (31.10) and its inverse (31.11) will belong to the group ${}^*\mathfrak{G}$.

Under the transformation (31.10) the components of affine connection ${}^\star\Gamma^\alpha_{\beta\gamma}$ of the representation $A^\star_{n+1}$ become ${}^\star C^\alpha_{\beta\gamma}$ in accordance with the equations

$$(31.12)\qquad {}^\star C^\alpha_{\beta\gamma}=\frac{\partial\eta^\alpha}{\partial x^\sigma}\left(\frac{\partial^2 x^\sigma}{\partial\eta^\beta\partial\eta^\gamma}+{}^\star\Gamma^\sigma_{\mu\nu}\frac{\partial x^\mu}{\partial\eta^\beta}\frac{\partial x^\nu}{\partial\eta^\gamma}\right).$$

When regard is taken of the form of the relations (31.10) and (31.11) we see from (31.12) that

$$(31.13)\qquad {}^\star C^\alpha_{\beta 0}={}^\star C^\alpha_{0\beta}=-\frac{\delta^\alpha_\beta}{n+1}.$$

Also it is seen that

$$(31.14)\qquad {}^\star C^i_{jk}=\frac{\partial\eta^i}{\partial x^m}\left(\frac{\partial^2 x^m}{\partial\eta^j\partial\eta^k}+{}^\star\Gamma^m_{\mu\nu}\frac{\partial x^\mu}{\partial\eta^j}\frac{\partial x^\nu}{\partial\eta^k}\right);$$

hence the identical relations

$$(31.15)\qquad {}^\star C^i_{jk}\eta^j\eta^k=0$$

follow in consequence of (31.4). Taking $i=k$ in (31.14) and summing on the indices i and k, we obtain

$$(31.16)\qquad {}^\star C^i_{ji}=\frac{\partial\log(x\eta)}{\partial\eta^j}-\frac{\partial x^0}{\partial\eta^j},$$

since $\Pi^i_{ji}=0$. Hence we can write

$$(31.17)\qquad {}^\star C^i_{ji}=\frac{\partial\phi}{\partial\eta^j},$$

where ϕ is an analytic function of the coordinates η^i.

As conditions of integrability of the equations (31.12) we can deduce

$$(31.18)\qquad {}^\star B^\sigma_{\mu\nu\xi}={}^\star D^\alpha_{\beta\gamma\delta}\frac{\partial\eta^\beta}{\partial x^\mu}\frac{\partial\eta^\gamma}{\partial x^\nu}\frac{\partial\eta^\delta}{\partial x^\xi}\frac{\partial x^\sigma}{\partial\eta^\alpha}$$

as in § 12; here ${}^\star B^\sigma_{\mu\nu\xi}$ denotes the components of the projective curvature tensor and the ${}^\star D^\alpha_{\beta\gamma\delta}$ stand for analogous expressions, namely

$$ {}^\star D^\alpha_{\beta\gamma\delta}=\frac{\partial{}^\star C^\alpha_{\beta\gamma}}{\partial\eta^\delta}-\frac{\partial{}^\star C^\alpha_{\beta\delta}}{\partial\eta^\gamma}+{}^\star C^\mu_{\beta\gamma}{}^\star C^\alpha_{\mu\delta}-{}^\star C^\mu_{\beta\delta}{}^\star C^\alpha_{\mu\gamma}.$$

Now the contracted components ${}^\star B^\alpha_{\beta\gamma\alpha}$ vanish identically (see § 18). On account of this, equations (31.18) show that the components ${}^\star D^\alpha_{\beta\gamma\alpha}=0$ likewise; from these equations we can deduce

$$(31.19)\qquad {}^\star C^0_{jk}+\left(\frac{n+1}{n-1}\right){}^\star C^m_{ja}{}^\star C^a_{mk}=\left(\frac{n+1}{n-1}\right)\left[\frac{\partial{}^\star C^m_{jk}}{\partial\eta^m}-\frac{\partial{}^\star C^m_{jm}}{\partial\eta^k}+{}^\star C^m_{jk}{}^\star C^a_{ma}\right].$$

Multiplying (31.19) by $\eta^j\eta^k$ and summing on the indices j and k, shows that

$$(31.20)\qquad \left[{}^\star C^0_{jk}+\left(\frac{n+1}{n-1}\right){}^\star C^m_{ja}{}^\star C^a_{mk}\right]\eta^j\eta^k=-\left(\frac{n+1}{n-1}\right)\left[\frac{\partial^2\phi}{\partial\eta^j\partial\eta^k}\eta^j\eta^k+2\frac{\partial\phi}{\partial\eta^j}\eta^j\right],$$

when use is made of (31.15), (31.17) and the set of equations which result from (31.15) by differentiation. Hence if we put

$$\left[{}^*C^0_{jk} + \left(\frac{n+1}{n-1}\right){}^*C^m_{ja}\,{}^*C^a_{mk}\right]\eta^j\eta^k = 0, \tag{31.21}$$

the equation resulting from (31.20), namely

$$\frac{\partial^2\phi}{\partial\eta^j\,\partial\eta^k}\,\eta^j\eta^k + 2\frac{\partial\phi}{\partial\eta^j}\,\eta^j = 0,$$

shows that $\phi=\text{const.}$; in fact we see by repeated differentiation of the above equations and evaluation at $\eta^i=0$, that the successive coefficients vanish in the power series expansion of the function ϕ. It follows then from (31.16) and (31.17) that the required condition (31.3) is satisfied; the function ω^0 must then be equal to $\log(x\eta)+p^0$ and the transformation (31.10) will belong to the group ${}^*\mathfrak{G}$.

The condition (31.21) leads us immediately to a suitable expression for the bracket in (31.5). Let us observe that we have

$$ {}^*C^0_{jk} = \frac{\partial^2 x^0}{\partial\eta^j\,\partial\eta^k} + {}^*\Gamma^0_{\mu\nu}\frac{\partial x^\mu}{\partial\eta^j}\frac{\partial x^\nu}{\partial\eta^k} + \frac{\partial\eta^0}{\partial x^i}\left(\frac{\partial^2 x^i}{\partial\eta^j\,\partial\eta^k} + {}^*\Gamma_{\mu\nu}\right. \tag{31.22}$$

in which we consider that

$$\frac{\partial\eta^0}{\partial x^i} = -\frac{\partial x^0}{\partial\eta^m}\frac{\partial\eta^m}{\partial x^i}.$$

When we make the substitutions (31.14) and (31.22) in the bracket expression in (31.21), this becomes

$${}^*C^0_{jk} + \left(\frac{n+1}{n-1}\right){}^*C^m_{ja}\,{}^*C^a_{mk} = \frac{\partial^2 x^0}{\partial\eta^j\,\partial\eta^k} + [jk], \tag{31.23}$$

where

$$\begin{aligned}[jk] = {} & {}^*\Gamma^0_{\mu\nu}\frac{\partial x^\mu}{\partial\eta^j}\frac{\partial x^\nu}{\partial\eta^k} - \frac{\partial x^0}{\partial\eta^a}\frac{\partial\eta^a}{\partial x^b}\,{}^*\Gamma^b_{\mu\nu}\frac{\partial x^\mu}{\partial\eta^j}\frac{\partial x^\nu}{\partial\eta^k} - \frac{\partial x^0}{\partial\eta^a}\frac{\partial\eta^a}{\partial x^b}\frac{\partial^2 x^b}{\partial\eta^j\,\partial\eta^k} \\ & + \left(\frac{n+1}{n-1}\right)\left\{\frac{\partial^2 x^d}{\partial\eta^a\,\partial\eta^k}\frac{\partial^2 x^c}{\partial\eta^b\,\partial\eta^j}\frac{\partial\eta^a}{\partial x^c}\frac{\partial\eta^b}{\partial x^d}\right. \\ & + {}^*\Gamma^d_{\sigma\tau}\frac{\partial^2 x^c}{\partial\eta^j\,\partial\eta^b}\frac{\partial\eta^a}{\partial x^c}\frac{\partial\eta^b}{\partial x^d}\frac{\partial x^\sigma}{\partial\eta^a}\frac{\partial x^\tau}{\partial\eta^k} \\ & + {}^*\Gamma^c_{\mu\nu}\frac{\partial^2 x^d}{\partial\eta^k\,\partial\eta^a}\frac{\partial\eta^a}{\partial x^c}\frac{\partial\eta^b}{\partial x^d}\frac{\partial x^\mu}{\partial\eta^j}\frac{\partial x^\nu}{\partial\eta^b} \\ & \left. + {}^*\Gamma^c_{\mu\nu}\,{}^*\Gamma^d_{\sigma\tau}\frac{\partial\eta^a}{\partial x^c}\frac{\partial\eta^b}{\partial x^d}\frac{\partial x^\mu}{\partial\eta^j}\frac{\partial x^\nu}{\partial\eta^b}\frac{\partial x^\sigma}{\partial\eta^a}\frac{\partial x^\tau}{\partial\eta^k}\right\},\end{aligned} \tag{31.24}$$

and the expression $[jk]$ in (31.5) is to be taken as the above expression (31.24). This gives us the following result.

Equations (31.4) *and* (31.5) *have a unique solution* (31.6) *satisfying the initial conditions* (31.7) *such that* (31.8) *constitutes a solution of the equations* (31.2).

The convergence of the power series expansions defining the functions ω^α in (31.6) can be proved in the following manner. Let us form the equations

$$(31.25)\qquad \begin{cases} (a) & \dfrac{\partial^2 X^i}{\partial\eta^j\partial\eta^k} - F^i_{\mu\nu}(X)\dfrac{\partial X^\mu}{\partial\eta^j}\dfrac{\partial X^\nu}{\partial\eta^k} = 0, \\ (b) & \dfrac{\partial^2 X^0}{\partial\eta^j\partial\eta^k} - \{jk\} = 0, \end{cases}$$

where the $\{jk\}$ are analytic functions of the quantities

$$(31.26)\qquad F^\alpha_{\beta\gamma};\quad \frac{\partial X^\alpha}{\partial\eta^j};\quad \frac{\partial^2 X^i}{\partial\eta^j\partial\eta^k}$$

which are taken to dominate the corresponding expressions $[jk]$ in (31.5). Since the second derivatives which occur in the expressions $\{jk\}$ are those which occur in (31.25 (a)), these derivatives can therefore be eliminated from the $\{jk\}$ by the substitution (31.25 (a)). We seek a solution $X^\alpha(\eta)$ of (31.25) satisfying the following conditions

$$(31.27)\qquad X^\alpha = p^\alpha,\quad \frac{\partial X^\alpha}{\partial\eta^j} = \delta^\alpha_j,\quad \text{when } \eta^j = 0.$$

Let us take

$$F^\alpha_{\beta\gamma} = F(X^0 + \ldots + X^n),$$

where the function F is analytic in the neighbourhood of $X^\alpha = p^\alpha$ and such as to dominate each of the functions $^\star\Gamma^\alpha_{\beta\gamma}$; let us also take each expression $\{jk\}$ to be equal to a function H of the sum of all the quantities (31.26). Then the integrability conditions of (31.25) are satisfied and the above solution $X^\alpha(\eta)$ of these equations exists. It is evident that the terms after the constant terms of the power series expansions of the functions $X^\alpha(\eta)$ dominate the corresponding terms of the expansions of the solution $x^\alpha(\eta)$ of the equations (31.4) and (31.5) with the result that these latter series converge.

The coordinates η^i defined by the transformation (31.8) will be called *projective normal coordinates* (3). We shall speak of the $n+1$ coordinates η^α defined by (31.10) as *projective normal coordinates for the affine representation* $A^\star_{n+1}$.

It can be shown that *when the coordinates x^i of the projective space P_n undergo an arbitrary transformation of the group $\mathfrak{G}$, the projective normal coordinates η^i which are determined by the x coordinate system and a point P undergo the linear fractional transformation*

$$(31.28)\qquad \eta^i = \frac{a^i_j\bar\eta^j}{1 + b_j\bar\eta^j},$$

where

$$a^i_j = \left(\frac{\partial x^i}{\partial\bar x^j}\right)_P,\qquad b_j = -\frac{1}{n+1}\left[\frac{\partial \log(x\bar x)}{\partial\bar x^j}\right]_P.$$

To prove this result let us add to the equations (31.28) the equation

$$(31.29)\qquad \eta^0 = \bar\eta^0 + \log(\eta\bar\eta),$$

where $(\eta\bar\eta)$ is the Jacobian determinant of the transformation (31.28), and then employ the method of § 29, i.e. we observe (1) that (31.28) and (31.29) satisfy the required initial conditions and are furthermore a solution of equations of the type (31.4) and (31.5) when referred to projective normal coordinates, and (2) that these latter equations possess a unique solution $\eta^\alpha(\bar\eta^0, \ldots, \bar\eta^n)$ satisfying the required initial conditions.

The equations (31.1), *together with the equations*

$$\Xi^i_{ji} = 0, \tag{31.30}$$

characterize the coordinates η^i *as projective normal coordinates.* To prove this we determine projective normal coordinates $\tilde{\eta}^i$ by the equations

$$\left(\frac{\partial^2 \eta^i}{\partial \tilde{\eta}^j \partial \tilde{\eta}^k} + {}^\star C^i_{\mu\nu} \frac{\partial \eta^\mu}{\partial \tilde{\eta}^j} \frac{\partial \eta^\nu}{\partial \tilde{\eta}^k}\right) \tilde{\eta}^j \tilde{\eta}^k = 0,$$

$$\left\{\frac{\partial^2 \eta^0}{\partial \tilde{\eta}^j \partial \tilde{\eta}^k} + (jk)\right\} \tilde{\eta}^j \tilde{\eta}^k = 0,$$

and the initial conditions,

$$\eta^\alpha = 0, \quad \frac{\partial \eta^\alpha}{\partial \tilde{\eta}^i} = \delta^\alpha_i, \quad \text{when} \quad \tilde{\eta}^i = 0;$$

here the $\tilde{C}^\alpha_{\beta\gamma}$ are determined from the components Ξ^i_{jk} by equations of the type (15.5) and the quantities (jk) are defined by equations of the type (31.24). The required solution of the above differential equations is uniquely determined and is in fact easily seen to be given by $\eta^i = \tilde{\eta}^i$, $\eta^0 = 0$ when use is made of (31.1) and (31.30). The coordinates η^i are therefore projective normal coordinates.

32. General theory of extension

Let the relative affine tensor T of weight W have components $T^{\alpha \dots \beta}_{\gamma \dots \delta}(x)$ with reference to the coordinates x^α and components $t^{\alpha \dots \beta}_{\gamma \dots \delta}(y)$ when referred to the system of affine normal coordinates y^α which are determined by the x coordinate system and a point P. We shall show that

$$T^{\alpha \dots \beta}_{\gamma \dots \delta, \epsilon} = \left(\frac{\partial t^{\alpha \dots \beta}_{\gamma \dots \delta}}{\partial y^\epsilon}\right)_0, \tag{32.1}$$

where the derivative is evaluated at the origin of normal coordinates, defines a set of functions $T^{\alpha \dots \beta}_{\gamma \dots \delta, \epsilon}(x)$ of the coordinates x^α which are the components of a relative tensor of weight W; this tensor is in fact identical with the first extension defined in § 14.

Denote by $\bar{t}^{\mu \dots \nu}_{\sigma \dots \tau}(\bar{y})$ the components of the tensor T with respect to the normal coordinates $\bar{y}^\alpha$ defined as in § 29; then we have

$$\bar{t}^{\mu \dots \nu}_{\sigma \dots \tau}(\bar{y}) = (y\bar{y})^W t^{\alpha \dots \beta}_{\gamma \dots \delta}(y) \frac{\partial \bar{y}^\mu}{\partial y^\alpha} \cdots \frac{\partial \bar{y}^\nu}{\partial y^\beta} \frac{\partial y^\gamma}{\partial \bar{y}^\sigma} \cdots \frac{\partial y^\delta}{\partial \bar{y}^\tau}. \tag{32.2}$$

In view of (29.3) the derivatives in (32.2) are constants and hence

$$\frac{\partial \bar{t}^{\mu \dots \nu}_{\tau \dots \tau}}{\partial \bar{y}^\omega} = (y\bar{y})^W \frac{\partial t^{\alpha \dots \beta}_{\gamma \dots \delta}}{\partial y^\epsilon} \frac{\partial \bar{y}^\mu}{\partial y^\alpha} \cdots \frac{\partial \bar{y}^\nu}{\partial y^\beta} \frac{\partial y^\gamma}{\partial \bar{y}^\sigma} \cdots \frac{\partial y^\delta}{\partial \bar{y}^\tau} \frac{\partial y^\epsilon}{\partial \bar{y}^\omega}.$$

Evaluating at the origin of normal coordinates we obtain

$$\bar{T}^{\mu \dots \nu}_{\sigma \dots \tau, \omega} = (x\bar{x})^W T^{\alpha \dots \beta}_{\gamma \dots \delta, \epsilon} \frac{\partial \bar{x}^\mu}{\partial x^\alpha} \cdots \frac{\partial \bar{x}^\nu}{\partial x^\beta} \frac{\partial x^\gamma}{\partial \bar{x}^\sigma} \cdots \frac{\partial x^\delta}{\partial \bar{x}^\tau} \frac{\partial x^\epsilon}{\partial \bar{x}^\omega},$$

which shows that the set of functions $T^{\alpha\dots\beta}_{\gamma\dots\delta,\epsilon}$ constitute the components of a relative tensor $T^\star$ of weight W.

To obtain the explicit formula involving the Γ's and their derivatives for the components of the tensor $T^\star$ we make use of the equations

$$t^{\alpha\dots\beta}_{\gamma\dots\delta}=(xy)^W\,T^{\mu\dots\nu}_{\sigma\dots\tau}\frac{\partial y^\alpha}{\partial x^\mu}\cdots\frac{\partial y^\beta}{\partial x^\nu}\frac{\partial x^\sigma}{\partial y^\gamma}\cdots\frac{\partial x^\tau}{\partial y^\delta}. \tag{32.3}$$

Differentiating these equations and evaluating at the origin of normal coordinates, we obtain*

$$\begin{aligned}T^{\alpha\dots\beta}_{\gamma\dots\delta,\epsilon}=\frac{\partial T^{\alpha\dots\beta}_{\gamma\dots\delta}}{\partial x^\epsilon}-T^{\alpha\dots\beta}_{k\dots\delta}\Gamma^k_{\gamma\epsilon}-\dots-T^{\alpha\dots\beta}_{\gamma\dots k}\Gamma^k_{\delta\epsilon}\\+T^{k\dots\beta}_{\gamma\dots\delta}\Gamma^\alpha_{k\epsilon}+\dots+T^{\alpha\dots k}_{\gamma\dots\delta}\Gamma^\beta_{k\epsilon}-WT^{\alpha\dots\beta}_{\gamma\dots\delta}\Gamma^k_{k\epsilon};\end{aligned} \tag{32.4}$$

this is the formula for the components of the first extension of the tensor T.

In a similar manner we can show that the equations

$$T^{\alpha\dots\beta}_{\gamma\dots\delta,\epsilon\dots\zeta}=\left(\frac{\partial^r t^{\alpha\dots\beta}_{\gamma\dots\delta}}{\partial y^\epsilon\dots\partial y^\zeta}\right)_0 \tag{32.5}$$

define a set of functions $T^{\alpha\dots\beta}_{\gamma\dots\delta,\epsilon\dots\zeta}$ of the coordinates x^α which constitute the components of a relative tensor of weight W. This relative tensor (4) will be called the *rth extension of the relative tensor* T provided that there are r indices $\epsilon, \dots, \zeta$. In case $r=1$ the extension reverts to the one previously considered.

From the definition by means of the equations (32.5) we see that the components $T^{\alpha\dots\beta}_{\gamma\dots\delta,\epsilon\dots\zeta}$ are symmetric in the indices $\epsilon, \dots, \zeta$. Thus

$$T^{\alpha\dots\beta}_{\gamma\dots\delta,\epsilon\dots\zeta}=T^{\alpha\dots\beta}_{\gamma\dots\delta,\xi\dots\pi},$$

where $\xi, \dots, \pi$ denotes any permutation of the indices $\epsilon, \dots, \zeta$.

The formulae for the components of the extension of the sum and product of two relative tensors are analogous to the formulae of ordinary differentiation of the sum and product of two functions; this follows directly from the definition of these components by means of (32.5).

General formulae of extension may be calculated by the same process as that employed in the derivation of (32.4). The formula for the components $T^{\alpha\dots\beta}_{\gamma\dots\delta,\epsilon\dots\zeta}$ of the rth extension of the tensor T of weight $W\neq 0$ involves the formulae for the first r extensions of T considered as a tensor of weight zero. We have

$$t^{\alpha\dots\beta}_{\gamma\dots\delta}=(xy)^W f^{\alpha\dots\beta}_{\gamma\dots\delta}, \tag{32.6}$$

where
$$f^{\alpha\dots\beta}_{\gamma\dots\delta}=T^{\mu\dots\nu}_{\sigma\dots\tau}\frac{\partial y^\alpha}{\partial x^\mu}\cdots\frac{\partial y^\beta}{\partial x^\nu}\frac{\partial x^\sigma}{\partial y^\gamma}\cdots\frac{\partial x^\tau}{\partial y^\delta}.$$

Differentiating (32.6) and evaluating at the origin of normal coordinates, we obtain

$$(32.7)\qquad T^{\alpha\dots\beta}_{\gamma\dots\delta,\,\epsilon\eta\dots\theta\zeta} = T^{\alpha\dots\beta}_{\gamma\dots\delta/\epsilon\eta\dots\theta\zeta} + S\,(\Delta^{W}_{\epsilon}\,T^{\alpha\dots\beta}_{\gamma\dots\delta/\eta\dots\theta\zeta}) + \cdots + S\,(\Delta^{W}_{\epsilon\eta\dots\theta}\,T^{\alpha\dots\beta}_{\gamma\dots\delta/\zeta}) + \Delta^{W}_{\epsilon\eta\dots\theta\zeta}\,T^{\alpha\dots\beta}_{\gamma\dots\delta},$$

where

$$T^{\alpha\dots}_{\gamma\dots} = \left(\frac{\partial^m f^{\alpha\dots\beta}_{\gamma\dots\delta}}{\partial y^{\mu}\dots\partial y^{\nu}}\right)_0,$$

$$\Delta^{W}_{\mu\dots\nu} = \left[\frac{\partial^m \log (xy)^W}{\partial y^{\mu}\dots\partial y^{\nu}}\right]_0,$$

and S ( ) denotes the sum of the different terms obtainable from the one in parenthesis by forming arbitrary combinations of the subscripts $\epsilon, \eta, \dots, \theta, \zeta$ which are distinct when account is taken of the symmetry in the added indices of differentiation. The expressions $T^{\alpha\dots\beta}_{\gamma\dots\delta/\mu\dots\nu}$, where there are m indices in the set $\mu, \dots, \nu$, are given by the formula for the mth extension of a tensor of weight zero having the components $T^{\alpha\dots\beta}_{\gamma\dots\delta}$; these expressions do not in general constitute the components of a tensor. The quantities $\Delta^{W}_{\mu\dots\nu}$ have the values

$$\Delta^{W}_{\mu} = -W\,\Gamma^{\alpha}_{\alpha\mu}$$
$$\Delta^{W}_{\mu\nu} = -W\,\Gamma^{\alpha}_{\alpha\mu\nu} + W\Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\alpha\nu} + W^2\,\Gamma^{\alpha}_{\alpha\mu}\Gamma^{\beta}_{\beta\nu},$$

in which the $\Gamma^{\alpha}_{\beta\gamma\delta}$, etc. are the functions of the $\Gamma^{\alpha}_{\beta\gamma}$ and their derivatives defined in § 3.

The formula for the components of any extension of a relative tensor T of weight W may be obtained by substituting the proper values of $T^{\alpha\dots\beta}_{\gamma\dots\delta/\mu\dots\nu}$ and $\Delta^{W}_{\mu\dots\nu}$ into (32.7). Thus we may write

$$T^{\alpha\dots\beta}_{\gamma\dots\delta,\,\epsilon} = T^{\alpha\dots\beta}_{\gamma\dots\delta/\epsilon} - W\,T^{\alpha\dots\beta}_{\gamma\dots\delta}\,\Gamma^{k}_{k\epsilon}$$

in place of the formula (32.4). In the following section a few particular expressions $T^{\alpha\dots\beta}_{\gamma\dots\delta/\mu\dots\nu}$ are given as formulae of the components of extensions of tensors of weight zero, and these when substituted into the equations (32.7) together with the values of $\Delta^{W}_{\mu\dots\nu}$ give complete formulae in terms of the $\Gamma^{\alpha}_{\beta\gamma}$ and their derivatives for the components of the extensions of certain relative tensors of weight W.

33. Some formulae of extension

In this section we shall give the explicit formulae for the components of certain extensions of the covariant vector and covariant tensor of the second order. Some of these formulae are used in our later work and the others may prove useful for purposes of reference. In these formulae S is used to indicate the sum of all distinct terms which can be formed from the one in the

parenthesis by replacing the given combination of the subscripts p, q or p, q, r or p, q, r, s by arbitrary combinations of these subscripts. Thus

$$S\left(\frac{\partial^2 T_i}{\partial x^\alpha \partial x^p}\Gamma^\alpha_{qr}\right)=\frac{\partial^2 T_i}{\partial x^\alpha \partial x^p}\Gamma^\alpha_{qr}+\frac{\partial^2 T_i}{\partial x^\alpha \partial x^q}\Gamma^\alpha_{pr}+\frac{\partial^2 T_i}{\partial x^\alpha \partial x^r}\Gamma^\alpha_{pq}.$$

The formulae in question are as follows:

$$T_{i,p}=\frac{\partial T_i}{\partial x^p}-T_\alpha\Gamma^\alpha_{ip}; \tag{33.1}$$

$$T_{i,pq}=\frac{\partial^2 T_i}{\partial x^p \partial x^q}-\frac{\partial T_i}{\partial x^\alpha}\Gamma^\alpha_{pq}-S\left(\frac{\partial T_\alpha}{\partial x^p}\Gamma^\alpha_{iq}\right)-T_\alpha\Gamma^\alpha_{ipq}; \tag{33.2}$$

$$\begin{aligned}T_{i,pqr}=&\frac{\partial^3 T_i}{\partial x^p \partial x^q \partial x^r}-S\left(\frac{\partial^2 T_i}{\partial x^\alpha \partial x^p}\Gamma^\alpha_{qr}\right)-S\left(\frac{\partial^2 T_\alpha}{\partial x^p \partial x^q}\Gamma^\alpha_{ir}\right)\\&+S\left(\frac{\partial T_\alpha}{\partial x^\beta}\Gamma^\alpha_{ip}\Gamma^\beta_{qr}\right)-S\left(\frac{\partial T_\alpha}{\partial x^p}\Gamma^\alpha_{iqr}\right)-\frac{\partial T_i}{\partial x^\alpha}\Gamma^\alpha_{pqr}-T_\alpha\Gamma^\alpha_{ipqr};\end{aligned} \tag{33.3}$$

$$\begin{aligned}T_{i,pqrs}=&\frac{\partial^4 T_i}{\partial x^p \partial x^q \partial x^r \partial x^s}-S\left(\frac{\partial^3 T_i}{\partial x^\alpha \partial x^p \partial x^q}\Gamma^\alpha_{rs}\right)-S\left(\frac{\partial^3 T_\alpha}{\partial x^p \partial x^q \partial x^r}\Gamma^\alpha_{is}\right)\\&+S\left(\frac{\partial^2 T_i}{\partial x^\alpha \partial x^\beta}\Gamma^\alpha_{pq}\Gamma^\beta_{rs}\right)+S\left(\frac{\partial^2 T_\alpha}{\partial x^\beta \partial x^p}\Gamma^\alpha_{iq}\Gamma^\beta_{rs}\right)\\&-S\left(\frac{\partial^2 T_i}{\partial x^\alpha \partial x^p}\Gamma^\alpha_{qrs}\right)-S\left(\frac{\partial^2 T_\alpha}{\partial x^p \partial x^q}\Gamma^\alpha_{irs}\right)\\&+S\left(\frac{\partial T_\alpha}{\partial x^\beta}\Gamma^\alpha_{ip}\Gamma^\beta_{qrs}\right)+S\left(\frac{\partial T_\alpha}{\partial x^\beta}\Gamma^\alpha_{ipq}\Gamma^\beta_{rs}\right)\\&-S\left(\frac{\partial T_\alpha}{\partial x^p}\Gamma^\alpha_{iqrs}\right)-\frac{\partial T_i}{\partial x^\alpha}\Gamma^\alpha_{pqrs}-T_\alpha\Gamma^\alpha_{ipqrs};\end{aligned} \tag{33.4}$$

$$T_{ij,p}=\frac{\partial T_{ij}}{\partial x^p}-T_{\alpha j}\Gamma^\alpha_{ip}-T_{i\alpha}\Gamma^\alpha_{jp}; \tag{33.5}$$

$$\begin{aligned}T_{ij,pq}=&\frac{\partial^2 T_{ij}}{\partial x^p \partial x^q}-\frac{\partial T_{ij}}{\partial x^\alpha}\Gamma^\alpha_{pq}-S\left(\frac{\partial T_{\alpha j}}{\partial x^p}\Gamma^\alpha_{iq}\right)-S\left(\frac{\partial T_{i\alpha}}{\partial x^p}\Gamma^\alpha_{jq}\right)\\&+S\left(T_{\alpha\beta}\Gamma^\alpha_{ip}\Gamma^\beta_{jq}\right)-T_{\alpha j}\Gamma^\alpha_{ipq}-T_{i\alpha}\Gamma^\alpha_{jpq};\end{aligned} \tag{33.6}$$

$$\begin{aligned}T_{ij,pqr}=&\frac{\partial^3 T_{ij}}{\partial x^p \partial x^q \partial x^r}\quad\left(\frac{\partial^2 T_{\alpha j}}{\partial x^p \partial x^q}\Gamma^\alpha_{ir}\right)-S\left(\frac{\partial^2 T_{i}}{\partial x^p \partial}\right.\\&-S\left(\frac{\partial^2 T_{ij}}{\partial x^\alpha \partial x^p}\Gamma^\alpha_{qr}\right)+S\left(\frac{\partial T_{\alpha j}}{\partial x^\beta}\Gamma^\alpha_{ip}\Gamma^\beta_{qr}\right)+S\left(\frac{\partial T_{i\alpha}}{\partial x^\beta}\Gamma^\alpha_{jp}\Gamma^\beta_{qr}\right)\\&+S\left(\frac{\partial T_{\alpha\beta}}{\partial x^p}\Gamma^\alpha_{iq}\Gamma^\beta_{jr}\right)-S\left(\frac{\partial T_{\alpha j}}{\partial x^p}\Gamma^\alpha_{iqr}\right)\\&-S\left(\frac{\partial T_{i\alpha}}{\partial x^p}\Gamma^\alpha_{jqr}\right)-\frac{\partial T_{ij}}{\partial x^\alpha}\Gamma^\alpha_{pqr}+S\left(T_{\alpha\beta}\Gamma^\alpha_{ip}\Gamma^\beta_{jqr}\right)\\&+S\left(T_{\alpha\beta}\Gamma^\alpha_{ipq}\Gamma^\beta_{jr}\right)-T_{\alpha j}\Gamma^\alpha_{ipqr}-T_{i\alpha}\Gamma^\alpha_{jpqr};\end{aligned} \tag{33.7}$$

$$
\begin{aligned}
(33.8)\qquad T_{ij,pqrs} = {} & \frac{\partial^4 T_{ij}}{\partial x^p\,\partial x^q\,\partial x^r\,\partial x^s} - S\left(\frac{\partial^3 T_{\alpha j}}{\partial x^p\,\partial x^q\,\partial x^r}\,\Gamma^\alpha_{is}\right) - S\left(\frac{\partial^3 T_{i\alpha}}{\partial x^p\,\partial x^q\,\partial x^r}\,\Gamma^\alpha_{js}\right) \\
& - S\left(\frac{\partial^3 T_{ij}}{\partial x^\alpha\,\partial x^p\,\partial x^q}\,\Gamma^\alpha_{rs}\right) + S\left(\frac{\partial^2 T_{\alpha\beta}}{\partial x^p\,\partial x^q}\,\Gamma^\alpha_{ir}\,\Gamma^\beta_{js}\right) \\
& + S\left(\frac{\partial^2 T_{\alpha j}}{\partial x^\beta\,\partial x^p}\,\Gamma^\alpha_{iq}\,\Gamma^\beta_{rs}\right) + S\left(\frac{\partial^2 T_{i\alpha}}{\partial x^\beta\,\partial x^p}\,\Gamma^\alpha_{jq}\,\Gamma^\beta_{rs}\right) \\
& - S\left(\frac{\partial^2 T_{i\alpha}}{\partial x^p\,\partial x^q}\,\Gamma^\alpha_{jrs}\right) - S\left(\frac{\partial^2 T_{ij}}{\partial x^\alpha\,\partial x^p}\,\Gamma^\alpha_{qrs}\right) \\
& + S\left(\frac{\partial^2 T_{ij}}{\partial x^\alpha\,\partial x^\beta}\,\Gamma^\alpha_{pq}\,\Gamma^\beta_{rs}\right) - S\left(\frac{\partial^2 T_{\alpha j}}{\partial x^p\,\partial x^q}\,\Gamma^\alpha_{irs}\right) - S\left(\frac{\partial T_{\alpha j}}{\partial x^p}\,\Gamma^\alpha_{iqrs}\right) \\
& + S\left(\frac{\partial T_{\alpha\beta}}{\partial x^p}\,\Gamma^\alpha_{iqr}\,\Gamma^\beta_{js}\right) + S\left(\frac{\partial T_{\alpha j}}{\partial x^\beta}\,\Gamma^\alpha_{ipq}\,\Gamma^\beta_{rs}\right) + S\left(\frac{\partial T_{\alpha\beta}}{\partial x^p}\,\Gamma^\alpha_{iq}\,\Gamma^\beta_{jrs}\right) \\
& - S\left(\frac{\partial T_{\alpha\beta}}{\partial x^\gamma}\,\Gamma^\alpha_{ip}\,\Gamma^\beta_{jq}\,\Gamma^\gamma_{rs}\right) + S\left(\frac{\partial T_{\alpha j}}{\partial x^\beta}\,\Gamma^\alpha_{ip}\,\Gamma^\beta_{qrs}\right) - S\left(\frac{\partial T_{i\alpha}}{\partial x^p}\,\Gamma^\alpha_{jqrs}\right) \\
& + S\left(\frac{\partial T_{i\alpha}}{\partial x^\beta}\,\Gamma^\alpha_{jpq}\,\Gamma^\beta_{rs}\right) + S\left(\frac{\partial T_{i\alpha}}{\partial x^\beta}\,\Gamma^\alpha_{jp}\,\Gamma^\beta_{qrs}\right) - \frac{\partial T_{ij}}{\partial x^\alpha}\,\Gamma^\alpha_{pqrs} \\
& + S\left(T_{\alpha\beta}\,\Gamma^\alpha_{ipqr}\,\Gamma^\beta_{js}\right) + S\left(T_{\alpha\beta}\,\Gamma^\alpha_{ip}\,\Gamma^\beta_{jqrs}\right) + S\left(T_{\alpha\beta}\,\Gamma^\alpha_{ipq}\,\Gamma^\beta_{jrs}\right) \\
& - T_{\alpha j}\,\Gamma^\alpha_{ipqrs} - T_{i\alpha}\,\Gamma^\alpha_{jpqrs}.
\end{aligned}
$$

34. Scalar differentiation in a space of distant parallelism

If we transform the components of a tensor T to a system of absolute normal coordinates and evaluate at the origin of this system, we obtain a set of quantities which are of the nature of *absolute scalars** with respect to transformations of the x^α coordinates. To prove this formally we shall find it convenient to denote the set of tensor components $T^{\alpha\dots\beta}_{\mu\dots\nu}(x)$ with respect to the x system by $t^{|i\dots j|}_{|k\dots m|}$ in the z coordinate system, i.e., we shall adopt the convention that in the z coordinate system, Latin letters when enclosed by $|\quad|$ correspond to indices of covariant or contravariant character. If we put

$$T^{i\dots j}_{k\dots m} = \left(t^{|i\dots j|}_{|k\dots m|}\right)_{z=0},$$

then

$$T^{i\dots j}_{k\dots m}(x) = \overline{T}^{i\dots j}_{k\dots m}(\overline{x})$$

in consequence of (30.16). The explicit expressions for these scalars are obtained by evaluating at $z^i = 0$ both members of the set of equations

$$(34.1)\qquad t^{|i\dots j|}_{|k\dots m|} = T^{\alpha\dots\beta}_{\mu\dots\nu}\,\frac{\partial x^\mu}{\partial z^k}\cdots\frac{\partial x^\nu}{\partial z^m}\,\frac{\partial z^i}{\partial x^\alpha}\cdots\frac{\partial z^j}{\partial x^\beta}.$$

* Absolute scalar is here used in the sense of *scalar function*; in this way we avoid the use of the more lengthy expression "component of an absolute scalar" which we would have to employ if we used the word scalar in its strict sense as a special case of the tensor defined in § 10.

This gives $$T^{i\ldots j}_{k\ldots m} = T^{\alpha\ldots\beta}_{\mu\ldots\nu} h^{\mu}_{k} \ldots h^{\nu}_{m} h^{i}_{\alpha} \ldots h^{j}_{\beta}.$$

Similarly, if we differentiate the components $t^{|i\ldots j|}_{|k\ldots m|}$ any number of times and evaluate at the origin of the z system, we obtain a set of quantities, namely

$$T^{i\ldots j}_{k\ldots m,\,p\ldots q} = \left(\frac{\partial^r t^{|i\ldots j|}_{|k\ldots m|}}{\partial z^p \ldots \partial z^q}\right)_{z=0},$$

each of which is an absolute scalar with respect to analytic transformations of the x^{α} coordinates. On account of this property the name *scalar derivatives* will be used to refer to these quantities (5). To derive the explicit formulae for the scalar derivatives $T^{i\ldots j}_{k\ldots m,\,p\ldots q}$ we have merely to differentiate (34.1) with respect to $z^p, \ldots, z^q$ and evaluate at the origin of the z system. For example, the first scalar derivative of the covariant vector with components T_{α} is given by the formula

$$T_{k,p} = \left(\frac{\partial T_{\beta}}{\partial x^{\gamma}} - T_{\alpha}\Lambda^{\alpha}_{\beta\gamma}\right) h^{\beta}_{k} h^{\gamma}_{p}.$$

When we make a transformation (6.5) of the components of the fundamental vectors, the above components $t^{|i\ldots j|}_{|k\ldots m|}$ become $\star t^{|r\ldots s|}_{|u\ldots v|}$ in accordance with the equations

$$(34.2) \qquad \star t^{|r\ldots s|}_{|u\ldots v|} a^{i}_{r} \ldots a^{j}_{s} = t^{|i\ldots j|}_{|k\ldots m|} a^{k}_{u} \ldots a^{m}_{v}.$$

Evaluating these equations at the origin of the absolute normal coordinate system, we have

$$(34.3) \qquad \star T^{r\ldots s}_{u\ldots v} a^{i}_{r} \ldots a^{j}_{s} = T^{i\ldots j}_{k\ldots m} a^{k}_{u} \ldots,$$

also differentiating (34.2) and evaluating at the origin gives

$$(34.4) \qquad \star T^{r\ldots s}_{u\ldots v,\,f\ldots g} a^{i}_{r} \ldots a^{j}_{s} = T^{i\ldots j}_{k\ldots m,\,p\ldots q} a^{k}_{u} \ldots a^{m}_{v} a^{p}_{f} \ldots a^{q}_{g}.$$

We can express the result implied by the equations (34.3) and (34.4) by saying that *the scalars $T^{i\ldots j}_{k\ldots m}$ and the scalar derivatives $T^{i\ldots j}_{k\ldots m,\,p\ldots q}$ constitute the components of tensors with respect to transformations of the fundamental vectors.*

The point of view adopted in the above work, in which extension, or more precisely scalar differentiation, is brought into relationship with the absolute normal coordinate system, necessarily depends on a symmetric connection; we have taken this as the connection with components $\Lambda^{\alpha}_{\beta\gamma}$. It would be possible also to define extensions or scalar derivatives in a metric space of distant parallelism, on the basis of the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ derived from the components $g_{\alpha\beta}$, and this method can likewise be brought into relationship with a system of local coordinates; in fact, it is only necessary to replace Postulate C of the Postulates of the Absolute Normal Coordinate System by a similar postulate on the geodesics of the space of distant parallelism (see § 30).

In addition to the above two methods there is the method of covariant differentiation based on the non-symmetric components $\Delta^{\alpha}_{\beta\gamma}$ previously considered in Chapter II.

It is possible to develop a process by which the above methods are brought into relationship with one another and which will, moreover, permit the ready construction of tensors depending on combinations of these methods. This process has its geometrical foundation in the study of those surfaces $x^\alpha = f^\alpha(u, v)$ which are defined as solutions of the systems of equations

$$
(34.5)\qquad \begin{cases} \dfrac{\partial^2 x^\alpha}{\partial u \partial v} + \Delta^\alpha_{\beta\gamma} \dfrac{\partial x^\beta}{\partial u} \dfrac{\partial x^\gamma}{\partial v} = 0, \\ \dfrac{\partial^2 x^\alpha}{\partial u^2} + \Gamma^\alpha_{\beta\gamma} \dfrac{\partial x^\beta}{\partial u} \dfrac{\partial x^\gamma}{\partial u} = 0 \quad (v=0), \\ \dfrac{\partial^2 x^\alpha}{\partial v^2} + \Lambda^\alpha_{\beta\gamma} \dfrac{\partial x^\beta}{\partial v} \dfrac{\partial x^\gamma}{\partial v} = 0 \quad (u=0), \end{cases}
$$

and so constitutes a generalization of the process of covariant differentiation or extension, as developed in § 32. In this way we are led to a set of relations $x^\alpha = g^\alpha(y, z)$ which for $z^i = \text{const.}$ denote a transformation to a system of coordinates y^i, and which for $y^i = \text{const.}$ denote a transformation to a system of coordinates z^i. If $t(y, z)$ represents the components of a tensor either in the y or z coordinate system, then

$$
\left(\frac{\partial^r t(y, z)}{\partial y^\alpha \ldots \partial y^\beta \partial z^\gamma \ldots \partial z^\delta} \right)_{y=z=0}
$$

defines, when considered as a function of the x^α coordinates, the components of a tensor in the x coordinate system. As an alternative method of procedure the system (34.5) can be replaced by the system

$$
(34.6)\qquad \begin{cases} \left(\dfrac{\partial^2 x^\alpha}{\partial y^j \partial z^k} + \Delta^\alpha_{\beta\gamma} \dfrac{\partial x^\beta}{\partial y^j} \dfrac{\partial x^\gamma}{\partial z^k} \right) y^j z^k = 0, \\ \left(\dfrac{\partial^2 x^\alpha}{\partial y^j \partial y^k} + \Gamma^\alpha_{\beta\gamma} \dfrac{\partial x^\beta}{\partial y^j} \dfrac{\partial x^\gamma}{\partial y^k} \right) y^j y^k = 0 \quad (z=0), \\ \left(\dfrac{\partial^2 x^\alpha}{\partial z^j \partial z^k} + \Lambda^\alpha_{\beta\gamma} \dfrac{\partial x^\beta}{\partial z^j} \dfrac{\partial x^\gamma}{\partial z^k} \right) z^j z^k = 0 \quad (y=0). \end{cases}
$$

The consideration of system (34.6) enables us, moreover, by imposing initial conditions corresponding to (30.5) and (30.6), to develop a theory of absolute scalars which is a generalization of the above treatment.

35. Differential invariants defined by means of normal coordinates. Normal tensors

If we differentiate (29.6) twice and then evaluate at the origin of normal coordinates, we obtain

$$
(35.1)\qquad C^\alpha_{\beta\gamma}(0) = 0,
$$

i.e. *the components of affine connection $C^\alpha_{\beta\gamma}$ in a system of affine normal coordinates vanish at the origin.* Hence the power series for these components takes the form

$$
(35.2)\qquad C^\alpha_{\beta\gamma} = A^\alpha_{\beta\gamma\delta} y^\delta + \frac{1}{2!} A^\alpha_{\beta\gamma\delta\epsilon} y^\delta y^\epsilon + \ldots,
$$

in which the A's are the derivatives of $C^\alpha_{\beta\gamma}$ evaluated at the origin, i.e.

$$
(35.3)\qquad A^\alpha_{\beta\gamma\delta\ldots\sigma} = \left(\frac{\partial^r C^\alpha_{\beta\gamma}}{\partial y^\delta \ldots \partial y^\sigma} \right)_0 .
$$

The equations (35.3) can be taken as defining $A^\alpha_{\beta\gamma\delta\ldots\epsilon}$ as a set of functions of the coordinates x^α in such a way that at any point P the quantities

$A^{\alpha}_{\beta\gamma\delta\ldots\sigma}$ are equal to the right members of (35.3) evaluated in the system of normal coordinates having P as origin. The functions so defined are the components of tensors. To prove this we consider the transformation equations (29.8) in which the first derivatives are constants and the second derivatives accordingly vanish, on account of the form of the relation (29.3). Repeated differentiation of (29.8), followed by evaluation at the origin of normal coordinates, then gives

$$\bar{A}^{\tau}_{\beta\gamma\delta\ldots\sigma}\frac{\partial x^{\alpha}}{\partial \bar{x}^{\tau}}=A^{\alpha}_{\mu\nu\xi\ldots\rho}\frac{\partial x^{\mu}}{\partial \bar{x}^{\beta}}\frac{\partial x^{\nu}}{\partial \bar{x}^{\gamma}}\frac{\partial x^{\xi}}{\partial \bar{x}^{\delta}}\cdots\frac{\partial x^{\rho}}{\partial \bar{x}^{\sigma}}, \tag{35.4}$$

when use is made of (29.2); the tensor character of the quantities $A^{\alpha}_{\beta\gamma\delta\ldots\sigma}$ follows from (35.4). We call these tensors the *affine normal tensors* on account of their definition in terms of the components of affine connection in normal coordinates (6).

The components of the affine normal tensors A are expressible in terms of the $\Gamma^{\alpha}_{\beta\gamma}$ and their derivatives; on this account these tensors become by definition tensor differential invariants of the affinely connected space. If we differentiate the equations

$$C^{\sigma}_{\beta\gamma}\frac{\partial x^{\alpha}}{\partial y^{\sigma}}=\frac{\partial^2 x^{\alpha}}{\partial y^{\beta}\partial y^{\gamma}}+\Gamma^{\alpha}_{\mu\nu}\frac{\partial x^{\mu}}{\partial y^{\beta}}\frac{\partial x^{\nu}}{\partial y^{\gamma}}, \tag{35.5}$$

we obtain

$$\frac{\partial C^{\sigma}_{\beta\gamma}}{\partial y^{\delta}}\frac{\partial x^{\alpha}}{\partial y^{\sigma}}+C^{\sigma}_{\beta\gamma}\frac{\partial^2 x^{\alpha}}{\partial y^{\sigma}\partial y^{\delta}}=\frac{\partial^3 x^{\alpha}}{\partial y^{\beta}\partial y^{\gamma}\partial y^{\delta}}+\frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\xi}}\frac{\partial x^{\mu}}{\partial y^{\beta}}\frac{\partial x^{\nu}}{\partial y^{\gamma}}\frac{\partial x^{\xi}}{\partial y^{\delta}}$$
$$+\Gamma^{\alpha}_{\mu\nu}\frac{\partial^2 x^{\mu}}{\partial y^{\beta}\partial y^{\delta}}\frac{\partial x^{\nu}}{\partial y^{\gamma}}+\Gamma^{\alpha}_{\mu\nu}\frac{\partial x^{\mu}}{\partial y^{\beta}}\frac{\partial^2 x^{\nu}}{\partial y^{\gamma}\partial y^{\delta}}. \tag{35.6}$$

Substituting the values of the partial derivatives of x^{α} with respect to the y's as computed from (3.2) for the origin of normal coordinates, we find

$$A^{\alpha}_{\beta\gamma\delta}=\frac{\partial \Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}}-\Gamma^{\alpha}_{\beta\gamma\delta}-\Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta}-\Gamma^{\alpha}_{\beta\mu}\Gamma^{\mu}_{\gamma\delta}. \tag{35.7}$$

If we differentiate (35.6) again, we obtain

$$A^{\alpha}_{\beta\gamma\delta\epsilon}=\frac{\partial^2 \Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}\partial x^{\epsilon}}-\Gamma^{\alpha}_{\beta\gamma\delta\epsilon}-\frac{\partial \Gamma^{\alpha}_{\beta\mu}}{\partial x^{\delta}}\Gamma^{\mu}_{\gamma\epsilon}-\frac{\partial \Gamma^{\alpha}_{\mu\gamma}}{\partial x^{\delta}}\Gamma^{\mu}_{\beta\epsilon}-\frac{\partial \Gamma^{\alpha}_{\beta\mu}}{\partial x^{\epsilon}}\Gamma^{\mu}_{\gamma\delta}$$
$$-\frac{\partial \Gamma^{\alpha}_{\mu\gamma}}{\partial x^{\epsilon}}\Gamma^{\mu}_{\beta\delta}-\frac{\partial \Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\mu}}\Gamma^{\mu}_{\delta\epsilon}-\Gamma^{\alpha}_{\beta\mu}\Gamma^{\mu}_{\gamma\delta\epsilon}-\Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta\epsilon}$$
$$+A^{\mu}_{\beta\gamma\delta}\Gamma^{\alpha}_{\mu\epsilon}+A^{\mu}_{\beta\gamma\epsilon}\Gamma^{\alpha}_{\mu\delta}+\Gamma^{\alpha}_{\mu\nu}\Gamma^{\mu}_{\beta\delta}\Gamma^{\nu}_{\gamma\epsilon}+\Gamma^{\alpha}_{\mu\nu}\Gamma^{\mu}_{\beta\epsilon}\Gamma^{\nu}_{\gamma\delta}.$$

It is evident that a continuation of this process will determine the explicit formula for the components of any other affine normal tensor A.

A similar process carried out in the affine representation $A^{\star}_{n+1}$ of the projective space P_n will lead to the definition of a class of tensors $A^{\star}$ with respect

to coordinate transformations of the group $^*\mathfrak{G}$. It is easily seen that *these tensors A^* are actually projective tensor differential invariants* as defined in § 18.

In an analogous manner we can define an infinite class of metric tensor differential invariants, called the *metric normal tensors*(7) whose components $g_{\alpha\beta,\gamma\ldots\delta}$ are given as functions of the coordinates x^α by the equations

$$g_{\alpha\beta,\gamma\ldots\delta}=\left(\frac{\partial^r\psi_{\alpha\beta}}{\partial y^\gamma\ldots\partial y^\delta}\right)_0, \tag{35.8}$$

where $\psi_{\alpha\beta}$ denotes the components of the fundamental metric tensor in the system of normal coordinates as in § 29. The method of § 32 can be applied to show that the quantities $g_{\alpha\beta,\gamma\ldots\delta}$ defined by (35.8) enjoy the tensor law of transformation, namely

$$\bar{g}_{\mu\nu,\sigma\ldots\tau}=g_{\alpha\beta,\gamma\ldots\delta}\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\cdots\frac{\partial x^\delta}{\partial \bar{x}^\tau},$$

and also to deduce the explicit formulae for these quantities. The first metric normal tensor having the components $g_{\alpha\beta,\gamma}$ is identical with the covariant derivative of the fundamental metric tensor and, as we know, vanishes. The formula for the components of the second metric normal tensor is

$$\begin{aligned}g_{\alpha\beta,\gamma\delta}=\frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma\partial x^\delta}-g_{\sigma\beta}\Gamma^\sigma_{\alpha\gamma\delta}-g_{\alpha\sigma}\Gamma^\sigma_{\beta\gamma\delta}-\frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\Gamma^\sigma_{\gamma\delta}-\frac{\partial g_{\sigma\beta}}{\partial x^\gamma}\Gamma^\sigma_{\alpha\delta}\\ -\frac{\partial g_{\alpha\sigma}}{\partial x^\gamma}\Gamma^\sigma_{\beta\delta}-\frac{\partial g_{\sigma\beta}}{\partial x^\delta}\Gamma^\sigma_{\alpha\gamma}-\frac{\partial g_{\alpha\sigma}}{\partial x^\delta}\Gamma^\sigma_{\beta\gamma}+g_{\sigma\tau}(\Gamma^\sigma_{\alpha\gamma}\Gamma^\tau_{\beta\delta}+\Gamma^\sigma_{\alpha\delta}\Gamma^\tau_{\beta\gamma}).\end{aligned} \tag{35.9}$$

In fact the formula (35.9) as well as the explicit formulae for the components of the third and fourth of the metric normal tensors can be obtained by replacing the components T_{ij} by the components of the fundamental metric tensor in the equations (33.6), (33.7) and (33.8) respectively.

We next consider the scalar derivatives of the fundamental vectors, using the covariant components h^i_α of these vectors as the basis of discussion. Let us denote these components by $A^i_{|j|}$ when referred to the z coordinate system. Then

$$A^i_{|j|}=h^i_\alpha\frac{\partial x^\alpha}{\partial z^j}. \tag{35.10}$$

In general, the scalar derivatives are defined by the equations

$$h^i_{j,k\ldots m}=\left(\frac{\partial^s A^i_{|j|}}{\partial z^k\ldots\partial z^m}\right)_{z=0}. \tag{35.11}$$

We note that

$$A^i_{|j|}=h^i_\alpha h^\alpha_j=\delta^i_j \quad (z^i=0); \tag{35.12}$$

also that

$$h^i_{j,k}=\frac{1}{2}\left(\frac{\partial h^i_\alpha}{\partial x^\beta}-\frac{\partial h^i_\beta}{\partial x^\alpha}\right)h^\alpha_j h^\beta_k, \tag{35.13}$$

and

$$(35.14)\qquad h^i_{j,kl}=\left(\frac{\partial^2 h^i_\alpha}{\partial x^\beta\,\partial x^\gamma}-\frac{\partial h^i_\alpha}{\partial x^\sigma}\Lambda^\sigma_{\beta\gamma}-\frac{\partial h^i_\sigma}{\partial x^\beta}\Lambda^\sigma_{\alpha\gamma}-\frac{\partial h^i_\sigma}{\partial x^\gamma}\Lambda^\sigma_{\alpha\beta}-h^i_\sigma\Lambda^\sigma_{\alpha\beta\gamma}\right)h^\alpha_j h^\beta_k h^\gamma_l,$$

use being made of (30.11). The special formulae (35.13) and (35.14) will be important in our later work.

Since any scalar derivatives $h^i_{j,k\ldots m}$ is expressible in terms of the h^i_α and their derivatives by formulae of the type (35.14), these quantities are scalar differential invariants of the space of distant parallelism.

When we make a transformation (6.5) of the components of the fundamental vectors, the components $A^i_{|j|}$ go over into a set of components $\star A^i_{|j|}$ referred to the $z_\star$ system, which are related to the $A^i_{|j|}$ by

$$(35.15)\qquad \star A^i_{|j|}a^p_i=A^p_{|q|}a^q_j;$$

this follows from (6.5), (30.16) and (35.10). Differentiating both members of (35.15) with respect to $z^k_\star, \ldots, z^m_\star$ and evaluating at the origin of the absolute normal system, we obtain

$$(35.16)\qquad \star h^i_{j,k\ldots m}a^p_i=h^p_{q,r\ldots t}a^q_j\ldots a^t_m.$$

These equations correspond to the equations (34.4) and enable us to say likewise that the *scalar derivatives* $h^p_{q,r\ldots t}$ *constitute the components of a tensor with respect to transformations of the fundamental vectors.* A similar discussion can of course be made on the basis of the contravariant components of the fundamental vectors.

36. A generalization of the affine normal tensors

By a method entirely analogous to that by which (29.6) was derived, we can deduce the identities

$$(36.1)\qquad \begin{cases} C^\alpha_{\beta\gamma\delta}y^\beta y^\gamma y^\delta=0,\\ \cdots\cdots\cdots\cdots\cdots\\ C^\alpha_{\beta\gamma\delta\ldots\epsilon}y^\beta y^\gamma y^\delta\ldots y^\epsilon=0,\\ \cdots\cdots\cdots\cdots\cdots\end{cases}$$

where the C's, which are symmetric in their lower indices, denote the quantities in affine normal coordinates which correspond to the functions $\Gamma^\alpha_{\beta\gamma\delta}$, etc. defined in § 3. Any set of functions $\Gamma^\alpha_{\beta\gamma\delta}$, $\Gamma^\alpha_{\beta\gamma\delta\epsilon}$, ... will be referred to as the components of a *generalized affine connection.* Repeated differentiation of the general equation (36.1), followed by evaluation at the origin of normal coordinates, shows that

$$(36.2)\qquad C^\alpha_{\beta\gamma\delta\ldots\epsilon}(0)=0,$$

i.e. *the quantities* $C^\alpha_{\beta\gamma\delta\ldots\epsilon}$ *vanish at the origin of the normal coordinate system;* the equations (36.2) are a generalization of the previous equation (35.1).

Now the components $C^\alpha_{\beta\gamma}$ are related to the $\Gamma^\alpha_{\beta\gamma}$ by (29.4) and there are similar equations of transformation for the other C's. Thus

$$C^\nu_{\alpha\beta\gamma}\frac{\partial x^\sigma}{\partial y^\nu}=\frac{\partial^3 x^\sigma}{\partial y^\alpha\partial y^\beta\partial y^\gamma}+\Gamma^\sigma_{\mu\nu\tau}\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\frac{\partial x^\tau}{\partial y^\gamma}$$
$$-\frac{\partial^2 x^\sigma}{\partial y^\mu\partial y^\alpha}C^\mu_{\beta\gamma}-\frac{\partial^2 x^\sigma}{\partial y^\mu\partial y^\beta}C^\mu_{\gamma\alpha}-\frac{\partial^2 x^\sigma}{\partial y^\mu\partial y^\gamma}C^\mu_{\alpha\beta}. \tag{36.3}$$

To deduce the equations of transformation of the general set of components $C^\alpha_{\beta\gamma\delta\ldots\epsilon}$, of which the above equations (36.3) are a special case, let us reconsider how the equations (29.4) are obtained on the basis of the principle of invariance of form of the equations of the paths

$$\frac{d^2x^\alpha}{ds^2}+\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}=0 \tag{36.4}$$

(see § 9). Differentiating the equations of transformation of the quantities dx^α/ds, namely

$$\frac{dx^\alpha}{ds}=\frac{\partial x^\alpha}{\partial y^\sigma}\frac{dy^\sigma}{ds}, \tag{36.5}$$

we have

$$\frac{d^2x^\alpha}{ds^2}=\frac{\partial^2 x^\alpha}{\partial y^\sigma\partial y^\tau}\frac{dy^\sigma}{ds}\frac{dy^\tau}{ds}+\frac{\partial x^\alpha}{\partial y^\sigma}\frac{d^2y^\sigma}{ds^2} \tag{36.6}$$

Substituting (36.5) and (36.6) into the equations (36.4), we find that the condition that (36.4) should remain invariant in form is that the $\Gamma^\alpha_{\beta\gamma}$ should transform by (29.4). Suppose now that it is possible to continue in this way and so get the equations of transformation of the first $p-1$ successive sets of quantities $\Gamma^\alpha_{\beta\gamma\ldots\mu}$ as the result of the condition that the equations (36.4) and

$$\frac{d^mx^\alpha}{ds^m}+\Gamma^\alpha_{\beta\gamma\ldots\mu}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}\cdots\frac{dx^\mu}{ds}=0, \tag{36.7}$$

for $m=3, \ldots, p-1$, should be invariant in form. In the y^α coordinates these equations then become

$$\frac{d^my^\alpha}{ds^m}+C^\alpha_{\beta\gamma\ldots\mu}\frac{dy^\beta}{ds}\frac{dy^\gamma}{ds}\cdots\frac{dy^\mu}{ds}=0, \tag{36.8}$$

where $m=2, \ldots, p-1$. To find the equations of transformation of the quantities $\Gamma^\alpha_{\beta\gamma\ldots\sigma}$ in (36.7) for $m=p$, we differentiate (36.6) repeatedly so as to obtain

$$\frac{d^px^\alpha}{ds^p}=\frac{\partial^p x^\alpha}{\partial y^\beta\ldots\partial y^\gamma}\frac{dy^\beta}{ds}\cdots\frac{dy^\gamma}{ds}+\frac{\partial x^\alpha}{\partial y^\nu}\frac{d^py^\nu}{ds^p}+\bigstar, \tag{36.9}$$

where the $\bigstar$ indicates terms involving d^my^α/ds^m for which $m<p$; these quantities d^my^α/ds^m are to be eliminated by means of (36.8) after which (36.9) is to be substituted into (36.7) for $m=p$. The requirement of invariance of form then gives

$$C^\nu_{\xi\ldots\pi}\frac{\partial x^\alpha}{\partial y^\nu}=\frac{\partial^p x^\alpha}{\partial y^\xi\ldots\partial y^\pi}+\Gamma^\alpha_{\beta\ldots\sigma}\frac{\partial x^\beta}{\partial y^\xi}\cdots\frac{\partial x^\sigma}{\partial y^\pi}+\bigstar, \tag{36.10}$$

where the ☆ denotes a polynomial composed of a sum of terms each of which involves a function $C^{\alpha}_{\beta\ldots\mu}$ with less than p subscripts and derivatives of the coordinate transformation of the second order at least. The equations of transformation of an arbitrary set of quantities $\Gamma^{\alpha}_{\beta\ldots\sigma}$ are thus given by (36.10).

On account of the above mentioned form of the equations (36.10) it is clear that under the transformation (29.3) relating the coordinates y^{α} and $\bar{y}^{\alpha}$ of the normal coordinate systems with origins at the same point P, the quantities $C^{\alpha}_{\beta\ldots\sigma}$ go into a set of quantities $\bar{C}^{\nu}_{\xi\ldots\pi}$ such that

$$\bar{C}^{\nu}_{\xi\ldots\pi}\frac{\partial y^{\alpha}}{\partial \bar{y}^{\nu}}=C^{\alpha}_{\beta\ldots\sigma}\frac{\partial y^{\beta}}{\partial \bar{y}^{\xi}}\cdots\frac{\partial y^{\sigma}}{\partial \bar{y}^{\pi}}.$$

Repeated differentiation of these equations, followed by evaluation at the origin of normal coordinates, shows that the quantities defined by

$$A^{\alpha}_{(\beta\ldots\sigma)\psi\ldots\omega}=\left(\frac{\partial^{r}C^{\alpha}_{\beta\ldots\sigma}}{\partial y^{\psi}\ldots\partial y^{\omega}}\right)_{0}$$

constitute the components of an affine tensor differential invariant; the explicit formulae for the components of these invariants in terms of the $\Gamma^{\alpha}_{\beta\gamma}$ and their derivatives can be obtained by differentiation of the equations (36.10) and evaluation at the origin of normal coordinates. For example, the formula

$$\begin{aligned}A^{\sigma}_{(\alpha\beta\gamma)\omega}=-\Gamma^{\sigma}_{\alpha\beta\gamma\omega}+\frac{\partial\Gamma^{\sigma}_{\alpha\beta\gamma}}{\partial x^{\omega}}-\Gamma^{\sigma}_{\mu\alpha\beta}\Gamma^{\mu}_{\gamma\omega}-\Gamma^{\sigma}_{\mu\alpha\gamma}\Gamma^{\mu}_{\beta\omega}\\ -\Gamma^{\sigma}_{\mu\beta\gamma}\Gamma^{\mu}_{\alpha\omega}+\Gamma^{\sigma}_{\mu\alpha}A^{\mu}_{\beta\gamma\omega}+\Gamma^{\sigma}_{\mu\beta}A^{\mu}_{\gamma\alpha\omega}+\Gamma^{\sigma}_{\mu\gamma}A^{\mu}_{\alpha\beta\omega}\end{aligned}$$

is obtained by a single differentiation of (36.3). These invariants are a generalization of the affine normal tensors and we shall accordingly refer to them as *generalized affine normal tensors*. In case the components $A^{\alpha}_{(\beta\ldots\sigma)\psi\ldots\omega}$ contain only two terms in the parenthesis, they are the components of a normal tensor and we shall then omit the parenthesis for simplicity.

37. Formulae of repeated extension

In this section we write down a few special formulae relating the components of tensors obtained by repeated extension to those obtained by the higher extensions and the affine normal tensors. In each case the formula is obtained by differentiating formulae of the type appearing in § 33 and evaluating at the origin of normal coordinates.

(37.1) $$T_{i,p,q}=T_{i,pq}-T_{\alpha}A^{\alpha}_{ipq};$$

(37.2) $$T_{i,p,q,r}=T_{i,pqr}-T_{i,\alpha}A^{\alpha}_{pqr}-T_{\alpha,p}A^{\alpha}_{iqr}$$

$$(37.3)\qquad \begin{aligned} T_{i,p,q,r,s} = T_{i,pqrs} &- T_{\alpha,pq}A^{\alpha}_{irs} - T_{\alpha,pr}A^{\alpha}_{iqs} - T_{\alpha,ps}A^{\alpha}_{iqr} - T_{\alpha,qr}A^{\alpha}_{ips} \\ &- T_{\alpha,qs}A^{\alpha}_{ipr} - T_{\alpha,rs}A^{\alpha}_{ipq} - T_{i,\alpha p}A^{\alpha}_{qrs} - T_{i,\alpha q}A^{\alpha}_{prs} \\ &- T_{i,\alpha r}A^{\alpha}_{pqs} - T_{i,\alpha s}A^{\alpha}_{pqr} - T_{\alpha,p}A^{\alpha}_{iqrs} - T_{\alpha,q}A^{\alpha}_{iprs} \\ &- T_{\alpha,r}A^{\alpha}_{ipqs} - T_{\alpha,s}A^{\alpha}_{ipqr} - T_{i,\alpha}A^{\alpha}_{pqrs} \\ &- T_{\alpha}(A^{\alpha}_{ipqrs} - A^{\alpha}_{\beta pr}A^{\beta}_{iqs} - A^{\alpha}_{\beta ps}A^{\beta}_{iqr} - A^{\alpha}_{\beta ir}A^{\beta}_{pqs} \\ &- A^{\alpha}_{\ldots}A^{\beta}_{\ldots} - A^{\alpha}_{\beta pq}A^{\beta}_{irs} - A^{\alpha}_{\beta iq}A^{\beta}_{prs} - A^{\alpha}_{\beta ip}A^{\beta}_{qrs}); \end{aligned}$$

$$(37.4)\qquad T_{ij,p,q} = T_{ij,pq} - T_{\alpha j}A^{\alpha}_{ipq} - T_{i\alpha}A^{\alpha}_{jpq}$$

$$(37.5)\qquad \begin{aligned} T_{ij,p,q,r} = T_{ij,pqr} &- T_{\alpha j,p}A^{\alpha}_{iqr} - T_{\alpha j,q}A^{\alpha}_{ipr} - T_{\alpha j,r}A^{\alpha}_{ipq} \\ &- T_{i\alpha,p}A^{\alpha}_{jqr} - T_{i\alpha,q}A^{\alpha}_{jpr} - T_{i\alpha,r}A^{\alpha}_{jpq} \\ &- T_{ij,\alpha}A^{\alpha}_{pqr} - T_{\alpha j}A^{\alpha}_{ipqr} - T_{i\alpha}A^{\alpha}_{jpqr}; \end{aligned}$$

$$(37.6)\qquad \begin{aligned} T_{ij,p,q,r,s} = T_{ij,pqrs} &- T_{\alpha j}A^{\alpha}_{ipqrs} - T_{i\alpha}A^{\alpha}_{jpqrs} - T_{ij,\alpha}A^{\alpha}_{pqrs} \\ &- T_{\alpha j,p}A^{\alpha}_{iqrs} - T_{\alpha j,q}A^{\alpha}_{iprs} - T_{\alpha j,r}A^{\alpha}_{ipqs} - T_{\alpha j,s}A^{\alpha}_{ipqr} \\ &- T_{i\alpha,p}A^{\alpha}_{jqrs} - T_{i\alpha,q}A^{\alpha}_{jprs} - T_{i\alpha,r}A^{\alpha}_{jpqs} - T_{i\alpha,s}A^{\alpha}_{jpqr} \\ &- T_{\alpha j,qr}A^{\alpha}_{ips} - T_{\alpha j,qs}A^{\alpha}_{ipr} - T_{\alpha j,rs}A^{\alpha}_{ipq} - T_{i\alpha,qr}A^{\alpha}_{jps} \\ &- T_{i\alpha,qs}A^{\alpha}_{jpr} - T_{i\alpha,rs}A^{\alpha}_{jpq} - T_{\alpha j,p,q}A^{\alpha}_{irs} - T_{\alpha j,p,r}A^{\alpha}_{iqs} \\ &- T_{\alpha j,p,s}A^{\alpha}_{iqr} - T_{i\alpha,p,q}A^{\alpha}_{jrs} - T_{i\alpha,p,r}A^{\alpha}_{jqs} - T_{i\alpha,p,s}A^{\alpha}_{jqr} \\ &- T_{ij,\alpha,q}A^{\alpha}_{prs} - T_{ij,\alpha,r}A^{\alpha}_{pqs} - T_{ij,\alpha,s}A^{\alpha}_{pqr} - T_{ij,\alpha,p}A^{\alpha}_{qrs}; \end{aligned}$$

$$(37.7)\qquad \begin{aligned} T^{lm\ldots n}_{ij\ldots k,p,q} = T^{lm\ldots n}_{ij\ldots k,pq} &+ T^{\alpha m\ldots n}_{ij\ldots k}A^{l}_{\alpha pq} + \ldots + T^{lm\ldots\alpha}_{ij\ldots k}A^{n}_{\alpha pq} \\ &- T^{lm\ldots n}_{\alpha j\ldots k}A^{\alpha}_{ipq} - \ldots - T^{lm\ldots n}_{ij\ldots\alpha}A^{\alpha}_{kpq}. \end{aligned}$$

The generalized affine normal tensors appear in some of the formulae of extension which generalize the above formulae. We here write down only the following four particular cases:

$$(37.8)\qquad T_{i,p,qr} = T_{i,pqr} - T_{\alpha,q}A^{\alpha}_{ipr} - T_{\alpha,r}A^{\alpha}_{ipq} - T_{\alpha}A^{\alpha}_{ipqr};$$

$$(37.9)\qquad T_{i,pq,r} = T_{i,pqr} - T_{i,\alpha}A^{\alpha}_{pqr} - T_{\alpha,p}A^{\alpha}_{iqr} - T_{\alpha,q}A^{\alpha}_{ipr} - T_{\alpha}A^{\alpha}_{(ipq)r};$$

$$(37.10)\qquad \begin{aligned} T_{ij,p,qr} = T_{ij,pqr} &- T_{\alpha j,q}A^{\alpha}_{ipr} - T_{\alpha j,r}A^{\alpha}_{ipq} - T_{i\alpha,q}A^{\alpha}_{jpr} \\ &- T_{i\alpha,r}A^{\alpha}_{jpq} - T_{\alpha j}A^{\alpha}_{ipqr} - T_{i\alpha}A^{\alpha}_{jpqr}; \end{aligned}$$

$$(37.11)\qquad \begin{aligned} T_{ij,pq,r} = T_{ij,pqr} &- T_{ij,\alpha}A^{\alpha}_{pqr} - T_{\alpha j,p}A^{\alpha}_{iqr} - T_{\alpha j,q}A^{\alpha}_{ipr} \\ &- T_{i\alpha,p}A^{\alpha}_{jqr} - T_{i\alpha,q}A^{\alpha}_{jpr} - T_{\alpha j}A^{\alpha}_{(ipq)r} - T_{i\alpha}A^{\alpha}_{(jpq)r}. \end{aligned}$$

38. A THEOREM ON THE AFFINE CONNECTION

Consider a set of functions $Z^{\nu}(x)$ defined by the power series expansion

$$(38.1)\qquad Z^{\nu}(x) = \sum_{s=2}^{\infty} (1/s!)\,(F^{\nu}_{\sigma_1\ldots\sigma_s})_p\,(x^{\sigma_1} - p^{\sigma_1})\ldots(x^{\sigma_s} - p^{\sigma_s}),$$

in which the constants p^{ν} are arbitrary and the coefficients are subject only to the condition that the series converges. The following theorem regarding

the series (38.1) in its relation to the components $\Gamma^{\nu}_{\alpha\beta}$ of the affine connection can be proved(8).

THEOREM. *Given an arbitrary affine connection and a set of arbitrary functions $Z^{\nu}(x)$ defined by* (38.1). *Then there exists a coordinate system x such that the components of the affine connection $\Gamma^{\nu}_{\alpha\beta}(x)$ and of the generalized connections $\Gamma^{\nu}_{\alpha\beta\ldots\epsilon}(x)$ at $x^{\nu}=p^{\nu}$ will take on the values of the coefficients of the power series expansion of the functions $Z^{\nu}(x)$.*

To prove this theorem we consider the system of normal coordinates y^{ν} determined by the x coordinate system and the point p, and denote the components of the affine connection in this system by the set of functions $C^{\nu}_{\alpha\beta}(y)$. Now construct the transformation

$$x^{\nu}=p^{\nu}+y^{\nu}-\sum_{s=2}^{\infty}(1/s!)\,(F^{\nu}_{\sigma_1\ldots\sigma_s})_p\, y^{\sigma_1}\ldots y^{\sigma_s}, \tag{38.2}$$

which is determined by the coefficients of the expansion of the functions Z^{ν}. All components $C^{\nu}_{\alpha\ldots\epsilon}(y)$ vanish at $y^{\nu}=0$ since the coordinates y^{ν} are normal coordinates. Hence if we evaluate both members of (36.10) at the origin of coordinates y^{ν}, we have

$$\Gamma^{\nu}_{\alpha\ldots\epsilon}(p)=-\left(\frac{\partial^p x^{\nu}}{\partial y^{\alpha}\ldots\partial y^{\epsilon}}\right)_0=(F^{\nu}_{\alpha\ldots\epsilon})_p$$

on account of (38.2). Hence the components of the connection $\Gamma^{\nu}_{\alpha\beta}(x)$ and of the generalized connections $\Gamma^{\nu}_{\alpha\beta\ldots\epsilon}(x)$ are equal to the coefficients of the power series (38.1).

The above theorem will later have an important application to the problem of showing the complete integrability of certain complicated systems of partial differential equations of tensor character.

39. Replacement theorems

Consider an affine tensor differential parameter of order (r, s) of the affine geometry of paths having the components

$$T^{\alpha\ldots\beta}_{\mu\ldots\nu}\left(\Gamma^{\alpha}_{\beta\gamma};\ \frac{\partial\Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}};\ \ldots;\ \frac{\partial^r\Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}\ldots\partial x^{\epsilon}};\ F^{(i)};\ \ldots;\ \frac{\partial^s F^{(i)}}{\partial x^{\psi}\ldots\partial x^{\omega}}\right) \tag{39.1}$$

(see § 15). If we transform the components of this parameter to a system of normal coordinates and then evaluate at the origin, we see that each component (39.1) is equal to the expression

$$T^{\alpha\ldots\beta}_{\mu\ldots\nu}(0;\ A^{\alpha}_{\beta\gamma\delta};\ \ldots;\ A^{\alpha}_{\beta\gamma\delta\ldots\epsilon};\ F^{(i)};\ \ldots;\ F^{(i)}_{\psi\ldots\omega}); \tag{39.2}$$

this gives us the following(9)

REPLACEMENT THEOREM. *The components* (39.1) *of any affine tensor differential parameter can be put into the form* (39.2) *by replacing the $\Gamma^{\alpha}_{\beta\gamma}$ by zero, the derivatives of the $\Gamma^{\alpha}_{\beta\gamma}$ by the corresponding components of a normal tensor and the derivatives of the scalars $F^{(i)}$ by the corresponding components of the extensions of these scalars.*

In particular the above theorem can be applied to projective tensor differential parameters and still more particularly to affine and projective tensor differential invariants (see §§ 11 and 18). For the case of the general affinely connected space where the components $L^{\alpha}_{\beta\gamma}$ of the affine connection are not symmetric in the indices β and γ, we have an analogous replacement theorem; here, however, the components $L^{\alpha}_{\beta\gamma}$ are replaced by $\Omega^{\alpha}_{\beta\gamma}$ and the derivatives of the $L^{\alpha}_{\beta\gamma}$ by the corresponding components of an affine normal tensor *plus* the corresponding components of the extension of the tensor Ω.

Analogous replacement theorems likewise hold for the tensor differential invariants and parameters of metric spaces, Weyl spaces and spaces of distant parallelism. Thus for the case of the metric space, the components $g_{\alpha\beta}$ remain unchanged, the first derivatives of these components are replaced by zero and higher derivatives by the corresponding components of the metric normal tensors; the statement regarding the replacement of the scalars $F^{(i)}$ and their derivatives remains, of course, the same as in the above italicized theorem.

An immediate consequence of the above replacement theorems is that there can exist no metric tensor differential invariant of order 1. Also there can exist no affine tensor differential invariant of order 0 in a space of symmetric affine connection; in case the connection is not symmetric, the components of such an invariant can depend at most on the components of the tensor Ω.

If we consider an absolute scalar differential invariant of the space of distant parallelism, for example

$$\Phi\left(h^i_\alpha;\ \frac{\partial h^i_\alpha}{\partial x^\beta};\ \ldots;\ \frac{\partial^r h^i_\alpha}{\partial x^\beta \ldots \partial x^\gamma}\right), \tag{39.3}$$

it would be natural to construct a replacement theorem by the aid of the absolute normal coordinate system rather than the affine normal coordinate system as in the preceding discussion. In this case the replacement theorem would consist of the statement that the expression (39.3) is identically equal to

$$\Phi\,(\delta^i_\alpha;\ h^i_{|\alpha,\beta|};\ \ldots;\ h^i_{|\alpha,\beta\ldots\gamma|}),$$

where the indices enclosed by the | | are of the nature of Latin indices in accordance with the convention of § 34; more generally, of course, if the expression Φ contains the scalars $F^{(i)}$ and their derivatives, these will be replaced by the corresponding absolute scalars.

REFERENCES

(1) See ref. (4), Chapter I.

(2) These postulates for the absolute normal coordinate system are essentially those given by T. Y. Thomas, "On the unified field theory I", *Proc. N.A.S.* **16** (1930), pp. 761–76. Such coordinates are not entirely new as H. Poincaré and J. A. Schouten introduced a type of normal coordinates in a group space, which may be considered as a space of distant parallelism. See H. Poincaré, "Quelques remarques sur les groupes continus", *Rend. Circ. Mat. Palermo*, **15** (1901), pp. 321–68, and J. A. Schouten, "Kontinuierliche Transformationsgruppen", *Math. Ann.* **102** (1929), pp. 244–72. See also A. D. Michal, "Scalar extensions of an orthogonal ennuple of vectors", *Amer. Math. Monthly*, **37** (1930), pp. 529–33.

(3) Projective normal coordinates were first introduced by O. Veblen and J. M. Thomas, "Projective normal coordinates for the geometry of paths", *Proc. N.A.S.* **11** (1925), pp. 204–7. See also L. P. Eisenhart, "Projective normal coordinates", *Proc. N.A.S.* **16** (1930), pp. 731–40. The method of treating the projective normal coordinates in § 31 is different from that used in the above papers.

(4) See O. Veblen and T. Y. Thomas, ref. (6), Chapter I. Additional material is contained in O. Veblen and T. Y. Thomas, "Extensions of relative tensors", *Trans. Amer. Math. Soc.* **26** (1924), pp. 373–7.

(5) See ref. (2).

(6) The affine normal tensors were first defined by O. Veblen, ref. (4), Chapter I.

(7) T. Y. Thomas, "The principle of equivalence in the theory of relativity", *Phil. Mag.* (6), **48** (1924), pp. 1056–68. Also see G. D. Birkhoff, *Relativity and Modern Physics* (Harvard Univ. Press, 1923), pp. 124 and 228.

(8) This theorem was given by T. Y. Thomas, "A theorem concerning the affine connection", *Amer. Journ. of Math.* **50** (1928), pp. 518–20.

(9) The replacement theorem first appeared in a treatment of the projective space of paths by T. Y. Thomas, ref. (2), Chapter III. It was later used by T. Y. Thomas and A. D. Michal, "Differential invariants of affinely connected manifolds", *Ann. of Math.* **28** (1927), pp. 199 and 200, and "Differential invariants of relative quadratic forms", *ibid.* p. 647; T. Y. Thomas, "Determination of affine and metric spaces by their differential invariants", *Math. Ann.* **101** (1929), p. 719.

CHAPTER VI

SPATIAL IDENTITIES

40. Complete sets of identities

A complete set of identities of the components of an invariant is a set of identities furnishing all the algebraic conditions on these components; hence every identity satisfied by the components of the invariant can be deduced from the identities of the complete set by algebraic processes(1). For example, the components $g_{\alpha\beta}$ of the fundamental metric tensor of a metric space satisfy the symmetry identities

$$g_{\alpha\beta}=g_{\beta\alpha}. \tag{40.1}$$

These constitute a complete set of identities of the components $g_{\alpha\beta}$ since at an arbitrary point P of the region $\mathscr{R}$ these components are obviously subject only to the conditions (40.1). Similarly the identities

$$\Gamma^{\alpha}_{\beta\gamma}=\Gamma^{\alpha}_{\gamma\beta} \tag{40.2}$$

are a complete set of identities of the components of affine connection of the geometry of paths (see § 2); in the case of the general affinely connected space the components $L^{\alpha}_{\beta\gamma}$ satisfy no identities of the above type since these components are evidently entirely arbitrary at an arbitrary point P of the region $\mathscr{R}$.

The determination of the complete sets of identities of the components of differential invariants of higher order is of course not as simple as in the case of the above illustrations. The methods of the following sections will, however, enable us to write down explicitly the complete sets of identities of the components of the most important tensor differential invariants of affine, metric, Weyl spaces and the space of distant parallelism. Since the treatment to be applied to the differential invariants of the Weyl space has seemed to be sufficiently illustrated by the methods employed for the affine and metric spaces and the space of distant parallelism, the discussion of the Weyl invariants has been omitted in this chapter.

41. Identities in the components of the normal tensors

We see from the equations (35.3) that the components of any normal tensor A are symmetric in their first two subscripts and also in the remaining ones, i.e.

$$A^{\alpha}_{\beta\gamma\delta\ldots\sigma}=A^{\alpha}_{\gamma\beta\delta\ldots\sigma},\quad A^{\alpha}_{\beta\gamma\delta\ldots\sigma}=A^{\alpha}_{\beta\gamma\mu\ldots\nu}, \tag{41.1}$$

where $\mu, \ldots, \nu$ denotes any permutation of the indices $\delta, \ldots, \sigma$. If we multiply both members of (35.2) by $y^{\beta}y^{\gamma}$ and sum on the indices β and γ, we see that

the components $A^{\alpha}_{\beta\gamma\delta\ldots\sigma}$ must in addition satisfy a set of identities of the form

$$S(A^{\alpha}_{\beta\gamma\delta\ldots\sigma})=0, \tag{41.2}$$

where S denotes the sum of the terms, not identical because of the symmetry identities (41.1), which are obtainable from the one in parenthesis by permutation of the indices $\beta, \ldots, \sigma$.

We shall show that the identities (41.1) and (41.2) constitute a complete set of identities for the components $A^{\alpha}_{\beta\gamma\delta\ldots\sigma}$ of the corresponding normal tensor A. Consider a sequence of sets of numbers

$$A^{\alpha}_{\beta\gamma\delta};\quad A^{\alpha}_{\beta\gamma\delta\epsilon};\quad A^{\alpha}_{\beta\gamma\delta\epsilon\eta};\tag{41.3}$$

which are chosen so as to satisfy the algebraic conditions (41.1) and (41.2) and also so that the series (35.2) converges; otherwise the sets of numbers A may be quite arbitrary. The functions $C^{\alpha}_{\beta\gamma}$ defined by (35.2) will then be the components of a symmetric affine connection in a system of normal coordinates y^{α}, since the functions $C^{\alpha}_{\beta\gamma}$ satisfy the symmetry conditions $C^{\alpha}_{\beta\gamma}=C^{\alpha}_{\gamma\beta}$ and the equations (29.6) which characterize the y^{α} as a set of normal coordinates. This is all that is essential for the completeness proof of the identities (41.1) and (41.2). However, it may be desirable to give the argument in more detail which we shall do in the following manner.

Let us associate the sequence of sets of numbers A with a point P of space covered by a system of coordinates x^{α}; let p^{α} be the coordinates of the point P. It is then possible to introduce a symmetric affine connection with components $\Gamma^{\alpha}_{\beta\gamma}(x)$ into the region $\mathscr{R}$ which forms the immediate neighbourhood of the point P, such that the components of the normal tensors A which arise assume the given values (41.3) at the point P. Such a set of functions $\Gamma^{\alpha}_{\beta\gamma}$ is easily seen to be given by the series

$$\Gamma^{\alpha}_{\beta\gamma}=A^{\alpha}_{\beta\gamma\delta}(x^{\delta}-p^{\delta})+\frac{1}{2!}A^{\alpha}_{\beta\gamma\delta\epsilon}(x^{\delta}-p^{\delta})(x^{\epsilon}-p^{\epsilon})+\ldots;\tag{41.4}$$

in fact the components $\Gamma^{\alpha}_{\beta\gamma}$ given by (41.4) are such that all the coefficients $\Gamma(p)$ of the series (3.2) vanish in view of (41.1) and (41.2), and hence the equations of transformation to a system of normal coordinates y^{α} become

$$x^{\alpha}=p^{\alpha}+y^{\alpha}.\tag{41.5}$$

Under the transformation (41.5) the components $\Gamma^{\alpha}_{\beta\gamma}$ become $C^{\alpha}_{\beta\gamma}$ as given by the series (35.2) in which the coefficients A are the given sets of numbers (41.3).

Now consider a space $\mathscr{T}$ having components of affine connection $\Gamma^{\alpha}_{\beta\gamma}$ referred to a system of coordinates x^{α}, and let

$$I(A^{\alpha}_{\beta\gamma\delta\ldots\sigma})=0\tag{41.6}$$

represent any set of identities satisfied by the components $A^{\alpha}_{\beta\gamma\delta\ldots\sigma}$ of the affine normal tensor A. Also consider a corresponding space $\overline{\mathscr{T}}$ referred to a

system of coordinates $\bar{x}^\alpha$, and having a symmetric affine connection with components $\bar{\Gamma}^\alpha_{\beta\gamma}(\bar{x})$ such that the components $\bar{A}^\alpha_{\beta\gamma\delta\ldots\sigma}(\bar{x})$ at the point P are subject only to the algebraic conditions (41.1) and (41.2). But the components $\bar{A}^\alpha_{\beta\gamma\delta\ldots\sigma}$ must likewise satisfy the set of identities (41.6), i.e.

$$I(\bar{A}^\alpha_{\beta\gamma\delta\ldots\sigma})=0,$$

and in particular must satisfy this set of identities at the point P. Hence the set of identities (41.6) can be satisfied by a set of functions $\bar{A}^\alpha_{\beta\gamma\delta\ldots\sigma}$ which are subject only to the algebraic conditions (41.1) and (41.2); *in other words, the identities* (41.1) *and* (41.2) *constitute a complete set of identities of the components* $A^\alpha_{\beta\gamma\delta\ldots\sigma}$ *of the affine normal tensor* A.

In particular let us observe that

$$A^\alpha_{\beta\gamma\delta}=A^\alpha_{\gamma\beta\delta},\quad A^\alpha_{\beta\gamma\delta}+A^\alpha_{\gamma\delta\beta}+A^\alpha_{\delta\beta\gamma}=0 \tag{41.7}$$

are a complete set of identities for the components of the first normal tensor A. Also a complete set of identities for the components of the second normal tensor A is given by the identities

$$A^\alpha_{\beta\gamma\delta\epsilon}=A^\alpha_{\gamma\beta\delta\epsilon}=A^\alpha_{\beta\gamma\epsilon\delta}, \tag{41.8}$$

$$A^\alpha_{\beta\gamma\delta\epsilon}+A^\alpha_{\beta\delta\epsilon\gamma}+A^\alpha_{\beta\epsilon\gamma\delta}+A^\alpha_{\gamma\delta\epsilon\beta}+A^\alpha_{\gamma\epsilon\beta\delta}+A^\alpha_{\delta\epsilon\beta\gamma}=0. \tag{41.9}$$

We shall have occasion to use the particular identities (41.7), ..., (41.9) in our later work.

We now consider the identities satisfied by the components of the metric normal tensors of order two and higher. It is seen immediately from the equations of definition (35.8) of the components of these tensors that we have

$$g_{\alpha\beta,\gamma\ldots\delta}=g_{\beta\alpha,\gamma\ldots\delta},\quad g_{\alpha\beta,\gamma\ldots\delta}=g_{\alpha\beta,\mu\ldots\nu}, \tag{41.10}$$

where $\mu, \ldots, \nu$ denotes any permutation of the indices $\gamma, \ldots, \delta$. To derive other identities satisfied by these components we have recourse to the equations (29.16). By repeated differentiation of (29.16) followed by evaluation at the origin of the normal coordinate system, we see that in addition to (41.10) we have the identities

$$S^*(g_{\alpha\beta,\gamma\ldots\delta})=0, \tag{41.11}$$

where S^* denotes the sum of all the terms which can be formed from the one in the parenthesis by cyclic permutation of the indices $\beta, \gamma, \ldots, \delta$.

The identities (41.10) *and* (41.11) *constitute a complete set of identities of the components* $g_{\alpha\beta,\gamma\ldots\delta}$ *of the metric normal tensor*. The proof of this statement is exactly analogous to the proof of the corresponding result for the affine normal tensors: we consider the sequence of sets of numbers

$$g_{\alpha\beta};\quad g_{\alpha\beta,\gamma\delta};\quad g_{\alpha\beta,\gamma\delta\epsilon}; \tag{41.12}$$

chosen so as to satisfy the algebraic conditions (40.1), (41.10) and (41.11), and also so that the series

$$\psi_{\alpha\beta}=g_{\alpha\beta}+\frac{1}{2!}g_{\alpha\beta,\gamma\delta}y^{\gamma}y^{\delta}+\dots \tag{41.13}$$

converges. Then the functions $\psi_{\alpha\beta}$ defined by the series (41.13) in which the coefficients g have the values of the sequence (41.12) satisfy the equations (29.16) which characterize the variables y^{α} as a set of normal coordinates; this suffices to prove the above result(1).

The identities (41.10) and (41.11) give

$$g_{\alpha\beta,\gamma\delta}=g_{\beta\alpha,\gamma\delta}=g_{\alpha\beta,\delta\gamma}, \tag{41.14}$$

$$g_{\alpha\beta,\gamma\delta}+g_{\alpha\gamma,\delta\beta}+g_{\alpha\delta,\beta\gamma}=0 \tag{41.15}$$

as the complete set of identities of the components $g_{\alpha\beta,\gamma\delta}$ of the metric normal tensor of the second order. We can derive an interesting set of identities in the components $g_{\alpha\beta,\gamma\delta}$ from the above identities (41.14) and (41.15). By an interchange of indices in (41.15) we have

$$g_{\gamma\delta,\alpha\beta}+g_{\gamma\alpha,\beta\delta}+g_{\gamma\beta,\delta\alpha}=0;$$

subtracting these equations from (41.15) we obtain

$$(g_{\alpha\beta,\gamma\delta}-g_{\gamma\delta,\alpha\beta})+(g_{\alpha\delta,\beta\gamma}-g_{\beta\gamma,\alpha\delta})=0,$$

or

$$(g_{\beta\alpha,\delta\gamma}-g_{\delta\gamma,\beta\alpha})+(g_{\beta\gamma,\alpha\delta}-g_{\alpha\delta,\beta\gamma})=0.$$

Finally subtracting these last two sets of equations, we obtain the identities

$$g_{\alpha\beta,\gamma\delta}=g_{\gamma\delta,\alpha\beta}. \tag{41.16}$$

42. Identities of the space of distant parallelism

By (35.11) the scalar derivative $h^i_{j,k\dots m}$ is symmetric in the indices $k, \dots, m$ so that

$$h^i_{j,k\dots m}=h^i_{j,p\dots q}, \tag{42.1}$$

where $p, \dots, q$ denotes any permutation of the indices $k, \dots, m$. To derive other identities satisfied by the $h^i_{j,k\dots m}$ we make use of the relations (30.18) which hold identically in the z coordinate system; by repeated differentiation of these equations and evaluation at the origin of the z system, we obtain

$$S^{\star}(h^i_{j,k\dots m})=0, \tag{42.2}$$

where the $S^{\star}$ is now used to stand for the summation of all the terms obtainable from the one in parenthesis by permuting the indices $j, k, \dots, m$ cyclically.

The identities (42.1) *and* (42.2) *constitute a complete set of identities of the scalar derivatives* $h^i_{j,k\dots m}$; the proof of this statement is analogous to the proofs of the corresponding results in § 41 and depends on the fact that the equations (30.18) characterize the variables z^i as absolute coordinates(2).

As special cases of the above result let us observe that

$$h^i_{j,k}+h^i_{k,j}=0 \tag{42.3}$$

and

$$h^i_{j,kl}=h^i_{j,lk}, \quad h^i_{j,kl}+h^i_{k,lj}+h^i_{l,jk}=0 \tag{42.4}$$

constitute the complete sets of identities of the quantities $h^i_{j,k}$ and $h^i_{j,kl}$, respectively.

43. Determination of the components of the normal tensors in terms of the components of their extensions

If we differentiate the equations (35.7) which express the components $A^\alpha_{\beta\gamma\delta}$ of the first normal tensor A in terms of the $\Gamma^\alpha_{\beta\gamma}$ and their first derivatives, and then evaluate at the origin of a system of normal coordinates, we obtain

$$A^\alpha_{\beta\gamma\delta,\epsilon}=A^\alpha_{\beta\gamma\delta\epsilon}-A^\alpha_{(\beta\gamma\delta)\epsilon}, \tag{43.1}$$

since the components of affine connection vanish at the origin of normal coordinates by (35.1). The components $A^\alpha_{(\beta\gamma\delta)\epsilon}$ of the generalized normal tensor can be eliminated from (43.1) by interchanging the indices γ and δ in these equations, which gives

$$A^\alpha_{\beta\gamma\delta\epsilon}=A^\alpha_{\beta\delta\gamma\epsilon}+A^\alpha_{\beta\gamma\delta,\epsilon}-A^\alpha_{\beta\delta\gamma,\epsilon}. \tag{43.2}$$

Now the identities (43.2), when combined with (41.9) in an obvious manner, yield the identities

$$6A^\alpha_{\beta\gamma\delta\epsilon}=4A^\alpha_{\beta\gamma\delta,\epsilon}+3A^\alpha_{\beta\gamma\epsilon,\delta}+A^\alpha_{\beta\delta\epsilon,\gamma}-A^\alpha_{\delta\beta\gamma,\epsilon}-A^\alpha_{\delta\epsilon\beta,\gamma}, \tag{43.3}$$

in the deduction of which use is made of (41.7) and (41.8).

We see that the symmetry identities (41.8) are not satisfied when the components in these equations are replaced by the corresponding right members of (43.3) provided that account is taken only of the identities (41.7) and identities derivable from these latter identities by extension, i.e.

$$\begin{cases} A^\alpha_{\beta\gamma\delta,\epsilon_1\ldots\epsilon_s}=A^\alpha_{\gamma\beta\delta,\epsilon_1\ldots\epsilon_s}=A^\alpha_{\beta\gamma\delta,\nu_1\ldots\nu_s}, \\ A^\alpha_{\beta\gamma\delta,\epsilon_1\ldots\epsilon_s}+A^\alpha_{\gamma\delta\beta,\epsilon_1\ldots\epsilon_s}+A^\alpha_{\delta\beta\gamma,\epsilon_1\ldots\epsilon_s}=0, \end{cases} \tag{43.4}$$

where $\nu_1,\ldots,\nu_s$ denotes any permutation of the indices $\epsilon_1,\ldots,\epsilon_s$. Thus when we interchange the indices β, γ in (43.3) and subtract, we obtain

$$A^\alpha_{\delta\epsilon\gamma,\beta}-A^\alpha_{\delta\gamma\epsilon,\beta}+A^\alpha_{\delta\beta\epsilon,\gamma}-A^\alpha_{\delta\epsilon\beta,\gamma}+A^\alpha_{\delta\gamma\beta,\epsilon}-A^\alpha_{\delta\beta\gamma,\epsilon}=0. \tag{43.5}$$

However, if we express $A^\alpha_{\beta\gamma\delta\epsilon}$ in terms of the components $A^\alpha_{\beta\gamma\delta,\epsilon}$ by means of a formula which can be derived from (43.3) by interchanging the indices β, γ in (43.3), adding to (43.3), and then interchanging the indices δ, ϵ in the equations so obtained and adding as before, the equations (41.8) will be

satisfied identically as a consequence of the equations (43.4) for $s=1$. We thus obtain

$$(43.6)\qquad 8A^{\alpha}_{\beta\gamma\delta\epsilon}=5A^{\alpha}_{\beta\gamma\delta,\,\epsilon}+5A^{\alpha}_{\beta\gamma\epsilon,\,\delta}-A^{\alpha}_{\delta\epsilon\beta,\,\gamma}-A^{\alpha}_{\delta\epsilon\gamma,\,\beta}$$

as the required equations. Also the formula (43.6) will satisfy (41.9) identically as a consequence of (43.4) for $s=1$. Perhaps the easiest way to see this is to imagine the left member of (41.9) replaced by the sum of the components $A^{\alpha}_{\beta\gamma\delta\epsilon}$ obtainable by taking *all* permutations of the subscripts β, γ, δ, ϵ which can be done as these equations are equivalent to (41.9) on account of (41.8). The substitution of (43.6) into the modified equations (41.9) will satisfy these equations in the required way inasmuch as each set of terms in (43.6) will lead to expressions which will vanish identically by (43.4) for $s=1$.

In view of the above property of the equations (43.6) we shall consider that the relationship between $A^{\alpha}_{\beta\gamma\delta\epsilon}$ and the components $A^{\alpha}_{\beta\gamma\delta,\,\epsilon}$ is expressed by these equations rather than by (43.3).

Let us now extend the result which we have just obtained. We differentiate the equations (35.7) repeatedly, say r times, and evaluate at the origin of normal coordinates. We then have

$$(43.7)\qquad A^{\alpha}_{\beta\gamma\delta,\,\epsilon_1\ldots\epsilon_r}=A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}-A^{\alpha}_{(\beta\gamma\delta)\epsilon_1\ldots\epsilon_r}+\star,$$

where the $\star$ indicates an expression in the quantities

$$A^{\alpha}_{\beta\gamma\delta};\quad A^{\alpha}_{\beta\gamma\delta\epsilon_1};\quad \ldots;\quad A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_{r-2}},$$

which is quadratic and homogeneous in these components. As before we interchange the indices γ and δ and subtract, which gives

$$A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}=A^{\alpha}_{\beta\delta\gamma\epsilon_1\ldots\epsilon_r}+A^{\alpha}_{\beta\gamma\delta,\,\epsilon_1\ldots\epsilon_r}-A^{\alpha}_{\beta\delta\gamma,\,\epsilon_1\ldots\epsilon_r}+\star.$$

On combining these latter equations with the identities (41.2) and making use of the symmetry identities (41.1), we find

$$(43.8)\qquad A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}=L(A^{\alpha}_{\beta\gamma\delta,\,\epsilon_1\ldots\epsilon_r})+\star,$$

in which L stands for a sum of the components indicated and which involves as free indices the indices $\alpha, \beta, \gamma, \delta, \epsilon_1, \ldots, \epsilon_r$ in the left member. The equations (43.8) can be made symmetric in the indices β, γ and $\delta, \epsilon_1, \ldots, \epsilon_r$ by the process of forming all permutations of these sets of indices in (43.8) and adding the resulting equations, as was done for the particular equations (43.3). It will accordingly be considered that the equations (43.8) satisfy these symmetry conditions.

By using (43.8) as a recurrence formula we can express the components $A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}$ in the form

$$(43.9)\qquad A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}=L\,(A^{\alpha}_{\beta\gamma\delta,\,\epsilon_1\ldots\epsilon_r})+M\,(A^{\alpha}_{\beta\gamma\delta};\ \ \ldots;\ \ A^{\alpha}_{\beta\gamma\delta,\,\epsilon_1\ldots\epsilon_{r-2}}),$$

where M denotes a sum of homogeneous polynomials in the components indicated; the expression M, just as the expression L, involves the indices α, β, γ, δ, ϵ_1, $\ldots$, ϵ_r as free indices. It is evident that the right members of

(43.9) are symmetric in the indices β, γ and also in the indices $\delta, \epsilon_1, \ldots, \epsilon_r$ on account of their method of formation by means of (43.8). The explicit form of the equations for a definite value of r may be capable of considerable simplification by use of equations of the type (43.4); allowing such substitutions in the right members of (43.9), we can say that these equations will satisfy identities of the type (41.1) when account is taken of (43.4) for $s \leqq r$.

It remains for us to see that the substitution (43.9) will satisfy the set of equations (41.2), which we will here write

$$\Sigma(A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}) = 0, \tag{43.10}$$

identically in the arguments which appear in the right members of (43.9) when account is taken only of the equations (41.7) and equations (43.4) for $s \leqq r$. Let us suppose, as for the special case already treated, that the identities (43.10) consist of the sum of the components $A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}$ that can be formed by taking *all* permutations of the subscripts $\beta, \gamma, \delta, \epsilon_1, \ldots, \epsilon_r$, which is equivalent to these equations as originally defined on account of the identities (41.1). Now replace the components in these equations (43.10) by their values as given by the right members of (43.9). Then it is obvious that the terms which arise from the expression L will vanish among themselves. Moreover it is just as obvious that the terms which arise from the expression M will likewise vanish in the required manner. We have in fact only to observe that any term in the expression M involves, as a factor, a component

$$A^{\mu}_{\xi\pi\rho} \quad \text{or} \quad A^{\mu}_{\xi\pi\rho,\omega_1\ldots\omega_v}, \tag{43.11}$$

where $v < r$ and $\xi, \pi, \rho, \omega_1, \ldots, \omega_v$ denote indices selected from the set $\beta, \gamma, \delta, \epsilon_1, \ldots, \epsilon_r$. Hence the term involving the component (43.11) gives rise to expressions which vanish in consequence of (41.7) or (43.4) for $s < r$ when (43.9) is substituted into (43.10).

We have shown that the equations (43.9), *which express the components of any normal tensor A of order $r+1$ for $r \geqq 1$ in terms of the components of the normal tensor $A^{\alpha}_{\beta\gamma\delta}$ and its extensions, are such that when the components $A^{\alpha}_{\beta\gamma\delta\ldots\epsilon}$ in the identities* (41.1) *and* (41.2) *are replaced by the corresponding right members of* (43.9) *the identities* (41.1) *and* (41.2) *will be satisfied as a consequence of* (41.7) *and the identities derivable from* (41.7) *by extension*, i.e. *identities of the type* (43.4) *for $s \leqq r$.*

Equations analogous to (43.9) can be derived for the metric case. For this purpose we consider the set of equations (35.9) which expresses the components $g_{\alpha\beta,\gamma\delta}$ in terms of the components $g_{\alpha\beta}$ and their first and second derivatives. Differentiating the equations (35.9) repeatedly and evaluating at the origin of normal coordinates, we obtain

$$g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_r} = g_{\alpha\beta,\gamma\delta\epsilon_1\ldots\epsilon_r} - g_{\sigma\beta}A^{\sigma}_{(\alpha\gamma\delta)\epsilon_1\ldots\epsilon_r} - g_{\alpha\sigma}A^{\sigma}_{(\beta\gamma\delta)\epsilon_1\ldots\epsilon_r} + \text{☆}, \tag{43.12}$$

where as before the ☆ denotes terms of lower order than those which have been written down explicitly. We must eliminate the components of the generalized normal tensor which occur in these equations. Interchange of the indices β and γ followed by subtraction of the resulting equations from (43.12) gives

$$g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r}-g_{\alpha\gamma,\beta\delta\epsilon_1\dots\epsilon_r}$$
$$=g_{\alpha\beta,\gamma\delta,\epsilon_1\dots\epsilon_r}-g_{\alpha\gamma,\beta\delta,\epsilon_1\dots\epsilon_r}+g_{\sigma\beta}A^{\sigma}_{(\alpha\gamma\delta)\epsilon_1\dots\epsilon_r}-g_{\sigma\gamma}A^{\sigma}_{(\alpha\beta\delta)\epsilon_1\dots\epsilon_r}+☆.$$

The components in question are then eliminated by interchanging the indices α and δ in the latter equations and subtracting, by which we obtain

$$(43.13)\quad g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r}-g_{\alpha\gamma,\beta\delta\epsilon_1\dots\epsilon_r}+g_{\delta\gamma,\beta\alpha\epsilon_1\dots\epsilon_r}-g_{\delta\beta,\gamma\alpha\epsilon_1\dots\epsilon_r}$$
$$=g_{\alpha\beta,\gamma\delta,\epsilon_1\dots\epsilon_r}-g_{\alpha\gamma,\beta\delta,\epsilon_1\dots\epsilon_r}+g_{\delta\gamma,\beta\alpha,\epsilon_1\dots\epsilon_r}-g_{\delta\beta,\gamma\alpha,\epsilon_1\dots\epsilon_r}+☆.$$

Let us now hold the indices α, β, γ fixed in the equations (43.13), permute the indices δ, ϵ_1, ..., ϵ_r cyclically and add the resulting equations. It will be convenient to use the symbol P to denote the operation of permuting the indices δ, ϵ_1, ..., ϵ_r cyclically while the remaining indices α, β, γ are fixed. We have

$$P\,(g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r}-g_{\alpha\gamma,\beta\delta\epsilon_1\dots\epsilon_r})=(r+1)\,(g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r}-g_{\alpha\gamma,\beta\delta\epsilon_1\dots\epsilon_r}),$$

and the two sets of equations:

$$P\,(g_{\delta\gamma,\beta\alpha\epsilon_1\dots\epsilon_r})=-g_{\beta\gamma,\alpha\delta\epsilon_1\dots\epsilon_r}-g_{\alpha\gamma,\beta\delta\epsilon_1\dots\epsilon_r},$$
$$P\,(g_{\delta\beta,\gamma\alpha\epsilon_1\dots\epsilon_r})=-g_{\gamma\beta,\alpha\delta\epsilon_1\dots\epsilon_r}-g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r},$$

where use is made of course of the identities (41.10) and (41.11). Hence the operation P on the equations (43.13) gives

$$(43.14)\quad (r+2)\,(g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r}-g_{\alpha\gamma,\beta\delta\epsilon_1\dots\epsilon_r})$$
$$=P\,(g_{\alpha\beta,\gamma\delta,\epsilon_1\dots\epsilon_r}+\cdots+g_{\delta\gamma,\beta\alpha,\epsilon_1\dots\epsilon_r}+☆),$$

in which the result of this operation is only indicated in the right member of these equations. We next define an operation P' analogous to P but different from P in that P' operates on the indices γ, δ, ϵ_1, ..., ϵ_r and in fact effects the sum of *all* permutations of the indices. With reference to the identities (41.10) and (41.11) we then find that

$$P'\,(g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r})=(r+2)\,!\,g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r},$$
$$P'\,(g_{\alpha\gamma,\beta\delta\epsilon_1\dots\epsilon_r})=-(r+1)\,!\,g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon_r};$$

and also that

$$P'P\,(g_{\alpha\beta,\gamma\delta,\epsilon_1\dots\epsilon_r}-g_{\alpha\gamma,\beta\delta,\epsilon_1\dots\epsilon_r}-g_{\delta\beta,\gamma\alpha,\epsilon_1\dots\epsilon_r}+g_{\delta\gamma,\beta\alpha,\epsilon_1\dots\epsilon_r})$$
$$=P'P\,(3g_{\alpha\beta,\gamma\delta,\epsilon_1\dots\epsilon_r}-g_{\delta\beta,\gamma\alpha,\epsilon_1\dots\epsilon_r}+g_{\gamma\beta,\alpha\delta,\epsilon_1\dots\epsilon_r})$$
$$=3P'P\,(g_{\alpha\beta,\gamma\delta,\epsilon_1\dots\epsilon_r})=6\,(r+1)\,!\,S\,(g_{\alpha\beta,\gamma\delta,\epsilon_1\dots\epsilon_r}),$$

where S denotes the sum of all the terms that can be obtained from the one in the parenthesis by allowing each pair of indices γ, δ, ϵ_1, ..., ϵ_r to have the position γ, δ in the term in the parenthesis. Hence we have

$$(r+2)\,(r+3)\,(r+1)\,!\,g_{\alpha\beta,\gamma\delta\epsilon_1\dots\epsilon}=6\,(r+1)\,!\,S\,(g_{\alpha\beta,\gamma\delta,\epsilon_1\dots\epsilon_r})+P'P\,(☆).$$

Now the last set of terms, i.e. the $\star$ terms in these equations, involve the components of extensions of $g_{\alpha\beta}$ of order r at most in the derivatives of the $g_{\alpha\beta}$. Hence these equations can be used as recurrence equations to express the $\star$ quantities in terms of the components of extensions of $g_{\alpha\beta,\gamma\delta}$ up to those of order $r-2$ inclusive. We can therefore write

$$P'P(\star)=(r+1)!\,Q\,(g_{\alpha\beta};\ \ldots;\ g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_{r-2}}),$$

in which the Q expressions are symmetric in the free indices α, β and γ, δ, ϵ_1, ..., ϵ_r which occur in them. It is easily seen that Q is a homogeneous polynomial in the components $g_{\alpha\beta,\gamma\delta}$ and their extensions with coefficients which depend on the components $g_{\alpha\beta}$. Hence

$$(43.15)\quad (r+2)(r+3)\,g_{\alpha\beta,\gamma\delta\epsilon_1\ldots\epsilon_r}=6S\,(g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_r})+Q\,(g_{\alpha\beta};\ \ldots;\ g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_{r-2}}).$$

The substitution (43.15) *for* $r\geqq 1$ *into the identities* (41.10) *and* (41.11) *satisfies these identities when account is taken only of the identities* (41.14) *and* (41.15) *and those derivable from these latter identities by extension,* i.e. *identities of the type*

$$(43.16)\quad \begin{cases} g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_s}=g_{\beta\alpha,\gamma\delta,\epsilon_1\ldots\epsilon_s}=g_{\alpha\beta,\delta\gamma,\epsilon_1\ldots\epsilon_s}=g_{\alpha\beta,\gamma\delta,\sigma_1\ldots\sigma_s},\\ g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_s}+g_{\alpha\gamma,\delta\beta,\epsilon_1\ldots\epsilon_s}+g_{\alpha\delta,\beta\gamma,\epsilon_1\ldots\epsilon_s}=0,\end{cases}$$

for $s\leqq r$, *where* σ_1, ..., σ_s *denotes any permutation of the indices* ϵ_1, ..., ϵ_s. That the identities (41.10) are satisfied by the right members of (43.15) is immediate, since the right members of (43.15) have been made symmetric in the indices α, β and γ, δ, ϵ_1, ..., ϵ_r by construction. The proof that the set of identities (41.11) is also satisfied can be made in a way analogous to that for the affine case already treated.

In the particular case $r=1$ for the equations (43.15) we observe that the $\star$ terms in (43.12) all vanish so that the Q expressions in (43.13) do not occur. Hence we have

$$(43.17)\qquad 2g_{\alpha\beta,\gamma\delta\epsilon}=g_{\alpha\beta,\gamma\delta,\epsilon}+g_{\alpha\beta,\delta\epsilon,\gamma}+g_{\alpha\beta,\epsilon\gamma,\delta},$$

and it is easily seen that these equations satisfy the identities (41.10) and (41.11) in the required manner(3).

44. Generalization of the preceding identities

From the method of obtaining the formula for any normal tensor $A^{\alpha}_{\beta\gamma\delta\ldots\epsilon}$ in terms of the $\Gamma^{\alpha}_{\beta\gamma}$ and their derivatives it is clear that we have

$$(44.1)\qquad A^{\alpha}_{\beta\gamma\delta\ldots\epsilon}=\frac{\partial^m\Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}\ldots\partial x^{\epsilon}}-\Gamma^{\alpha}_{\beta\gamma\delta\ldots\epsilon}+\star,$$

where the $\star$ terms represent derivatives of $\Gamma^{\alpha}_{\beta\gamma}$ of order lower than m. Differentiation of these equations and evaluation at the origin of normal coordinates give

$$(44.2)\qquad A^{\alpha}_{\beta\gamma\delta\ldots\epsilon,\sigma_1\ldots\sigma_r}=A^{\alpha}_{\beta\gamma\delta\ldots\epsilon\sigma_1\ldots\sigma_r}-A^{\alpha}_{(\beta\gamma\delta\ldots\epsilon)\sigma_1\ldots\sigma_r}+\star,$$

where the $\star$ terms now stand for an expression in the components of normal tensors of the following set:

$$(44.3)\qquad A^{\alpha}_{\beta\gamma\delta};\ A^{\alpha}_{\beta\gamma\delta\eta};\ \ldots;\ A^{\alpha}_{\beta\gamma\delta\ldots\epsilon\sigma_1\ldots\sigma_{r-2}}.$$

By operating on the equations (44.2) in exactly the same way as we have operated on (43.1), whereby use is made of the identities (41.1) and (41.2), we can obtain

$$A^{\alpha}_{\beta\gamma\delta\ldots\epsilon\sigma_1\ldots\sigma_r} = L(A^{\alpha}_{\beta\gamma\delta\ldots\epsilon,\sigma_1\ldots\sigma_r}) + ☆, \tag{44.4}$$

where as before L denotes a sum of the components of the tensor appearing as its argument. It is to be supposed that the right members of (44.4) have been made symmetric in the free indices β, γ and $\delta, \ldots, \epsilon, \sigma_1, \ldots, \sigma_r$ which occur in these equations. Using (44.4) as recurrence equations, we have

$$A^{\alpha}_{\beta\gamma\delta\ldots\epsilon\sigma_1\ldots\sigma_r} = L(A^{\alpha}_{\beta\gamma\delta\ldots\epsilon,\sigma_1\ldots\sigma_r}) + M, \tag{44.5}$$

in which the arguments involved in the expressions M are indicated by the equations

$$M = M(A^{\alpha}_{\beta\gamma\delta};\ \ldots;\ A^{\alpha}_{\beta\gamma\delta\ldots\epsilon};\ A^{\alpha}_{\beta\gamma\delta\ldots\epsilon,\sigma_1};\ \ldots;\ A^{\alpha}_{\beta\gamma\delta\ldots\epsilon,\sigma_1\ldots\sigma_{r-2}}).$$

It is evident by its method of construction that M is composed of a sum of homogeneous polynomials of its arguments.

If the right members of (44.5) *are substituted into the identities* (41.1) *and* (41.2) *in place of the normal tensors which occur in* (41.1) *and* (41.2), *these identities will be satisfied when account is taken only of the identities satisfied by the normal tensors of the set*

$$A^{\alpha}_{\beta\gamma\delta};\qquad A^{\alpha}_{\beta\gamma\delta\ldots\epsilon},$$

and those which can be obtained by extension from the identities (41.1) *and* (41.2) *satisfied by the normal tensor* $A^{\alpha}_{\beta\gamma\delta\ldots\epsilon}$.

The generalization of the identities (43.15) can be made in an analogous manner. We first write down the equations

$$g_{\alpha\beta,\gamma\ldots\epsilon} = \frac{\partial^m g_{\alpha\beta}}{\partial x^\gamma\ldots\partial x^\epsilon} - g_{\tau\beta}\Gamma^\tau_{\alpha\gamma\ldots\epsilon} - g_{\alpha\tau}\Gamma^\tau_{\beta\gamma\ldots\epsilon} + ☆, \tag{44.6}$$

where the ☆ denotes derivatives of $g_{\alpha\beta}$ of lower order than m, differentiate repeatedly and evaluate at the origin of normal coordinates. This gives

$$g_{\alpha\beta,\gamma\ldots\epsilon,\sigma_1\ldots\sigma_r} = g_{\alpha\beta,\gamma\ldots\epsilon\sigma_1\ldots\sigma_r} - g_{\tau\beta}A^\tau_{(\alpha\gamma\ldots\epsilon)\sigma_1\ldots\sigma_r} - g_{\alpha\tau}A^\tau_{(\beta\gamma\ldots\epsilon)\sigma_1\ldots\sigma_r} + ☆, \tag{44.7}$$

where the ☆ flow denotes the sum of a set of homogeneous polynomials in the components of the set

$$g_{\alpha\beta};\ g_{\alpha\beta,\gamma\delta};\ \ldots;\ g_{\alpha\beta,\gamma\ldots\epsilon};\ g_{\alpha\beta,\gamma\ldots\epsilon\sigma_1};\ \ldots;\ g_{\alpha\beta,\gamma\ldots\epsilon\sigma_1\ldots\sigma_{r-2}}.$$

The equations (44.7) are now to be treated in the same way as we have treated (43.12). We then obtain an identity of the following general form

$$g_{\alpha\beta,\gamma\ldots\epsilon\sigma_1\ldots\sigma_r} = L(g_{\alpha\beta,\gamma\ldots\epsilon,\sigma_1\ldots\sigma_r}) + Q, \tag{44.8}$$

where $$Q = Q(g_{\alpha\beta};\ \ldots;\ g_{\alpha\beta,\gamma\ldots\epsilon};\ g_{\alpha\beta,\gamma\ldots\epsilon,\sigma_1};\ g_{\alpha\beta,\gamma\ldots\epsilon,\sigma_1\ldots\sigma_{r-2}}).$$

In (44.8) the expressions L and Q denote, respectively, a simple sum and the sum of a number of homogeneous polynomials of their arguments.

The substitution (44.8) *into the identities* (41.10) *and* (41.11) *will satisfy these identities when account is taken only of the identities satisfied by the components of the set*

$$g_{\alpha\beta};\ g_{\alpha\beta,\gamma\delta};\ \ldots;\ g_{\alpha\beta,\gamma\ldots\epsilon},$$

and those identities which are derivable by extension from the identities (41.10) *and* (41.11) *satisfied by the components* $g_{\alpha\beta,\gamma\ldots\epsilon}$.

The proofs of the above italicized statements are analogous to the proofs of the corresponding results in § 43.

45. Space determination by tensor invariants

The following theorem expresses an application of the identities (43.9) and (43.15). It follows directly from these identities by putting the components of the extensions of the normal tensors which appear in them equal to the derivatives of the affine connection and fundamental metric tensor, respectively, at the origin of a system of normal coordinates.

THEOREM A. *Given a set of variables y^α to be considered as the coordinates of a system of normal coordinates and a set of functions $A^\alpha_{\beta\gamma\delta}(y)$ of these variables which are analytic in the neighbourhood of the origin $y^\alpha = 0$. Then there is uniquely determined by* (43.9) *a sequence of sets of constants*

$$\text{(I)} \qquad (A^\alpha_{\beta\gamma\delta\epsilon_1})_0; \ (A^\alpha_{\beta\gamma\delta\epsilon_1\epsilon_2})_0; \ \cdots$$

associated with the origin. Similarly the set of constants $(g_{\alpha\beta})_0$ and the set of functions $g_{\alpha\beta,\gamma\delta}(y)$ analytic in the neighbourhood of $y^\alpha = 0$ uniquely determine by (43.15) *the sequence of sets of constants*

$$\text{(II)} \qquad (g_{\alpha\beta,\gamma\delta\epsilon_1})_0; \ (g_{\alpha\beta,\gamma\delta\epsilon_1\epsilon_2})_0; \ \cdots$$

which are associated with the origin of the y^α coordinates.

The extension of this theorem can be made immediately by taking account of the identities (44.5) and (44.8).

THEOREM B. (1) *The sets of constants $(A^\alpha_{\beta\gamma\delta})_0; \ldots; (A^\alpha_{\beta\gamma\delta\ldots\epsilon})_0$ associated with the origin of a system of normal coordinates y^α, and the set of functions $A^\alpha_{\beta\gamma\delta\ldots\epsilon\eta}(y)$ analytic in the neighbourhood of the origin, uniquely determine by* (44.5) *the sequence of sets of constants*

$$\text{(III)} \qquad (A^\alpha_{\beta\gamma\delta\ldots\eta\sigma_1})_0; \ (A^\alpha_{\beta\gamma\delta\ldots\eta\sigma_1\sigma_2})_0; \ \cdots$$

associated with the origin. (2) *The sets of constants $(g_{\alpha\beta})_0; (g_{\alpha\beta,\gamma\delta})_0; \cdots; (g_{\alpha\beta,\gamma\delta\ldots\eta})_0$ and the set of functions $g_{\alpha\beta,\gamma\delta\ldots\eta\epsilon}(y)$ analytic in the neighbourhood of the origin $y^\alpha = 0$ uniquely determine by* (44.8) *the sequence of sets of constants*

$$\text{(IV)} \qquad (g_{\alpha\beta,\gamma\delta\ldots\epsilon\sigma_1})_0; \ (g_{\alpha\beta,\gamma\delta\ldots\epsilon\sigma_1\sigma_2})_0; \ \cdots$$

associated with the origin.

The discussion of § 44 shows that if the given quantities in the statement of Theorem B satisfy the complete sets of identities of the components of the corresponding tensor invariants, the derived quantities (III) and (IV) will do so likewise.

In order to state the general theorem with respect to any system of coordinates x^ν let us consider the sequence of sets of constants

$$\Gamma^\nu_{\sigma_1\sigma_2}(p); \ \Gamma^\nu_{\sigma_1\sigma_2\sigma_3}(p); \ \ldots,$$

which are to be thought of as associated with the point P having coordinates $x^\nu = p^\nu$. The constants $\Gamma(p)$ will be assumed to be symmetric in their lower indices $\sigma_1, \ldots, \sigma_s$ but otherwise are to be considered as arbitrary. Since the constants $\Gamma(p)$ determine the series

$$Z^\nu = \sum_{s=2}^{\infty} \frac{1}{s!} \Gamma^\nu_{\sigma_1\ldots\sigma_s}(p)(x^{\sigma_1} - p^{\sigma_1}) \ldots (x^{\sigma_s} - p^{\sigma_s}), \tag{45.1}$$

they can conveniently be supposed to be determined by this series (see § 38). When we observe that the rth extension of any tensor is given by an expression involving partial derivatives of the components of the tensor up to and including those of order r, and the quantities $\Gamma^\nu_{\sigma_1\ldots\sigma_s}$ where s takes on the values $2, \ldots, r+1$, we are enabled to generalize Theorem B to arbitrary coordinates in the following manner:

THEOREM C. *If the quantities in the statement of Theorem B are given at the point P having coordinates $x^\nu = p^\nu$ and in the neighbourhood of this point as functions of the variables x^ν and if in addition the series* (45.1) *are given, then the sets of constants* (III) *and* (IV), *as the case may be, are uniquely determined, where now these constants are to be thought of as associated with the point P.*

The given quantities in the statement of the above theorems together with the derived constants of the above sequences determine the series

$$C^{\alpha}_{\beta\gamma} = (A^{\alpha}_{\beta\gamma\delta})_0\, y^{\delta} + \frac{1}{2\,!}(A^{\alpha}_{\beta\gamma\delta\epsilon_1})_0\, y^{\delta} y^{\epsilon_1} + \dots \tag{45.2}$$

and

$$\psi_{\alpha\beta} = (g_{\alpha\beta})_0 + \frac{1}{2\,!}(g_{\alpha\beta,\,\gamma\delta})_0\, y^{\gamma} y^{\delta} + \frac{1}{3\,!}(g_{\alpha\beta,\,\gamma\delta\epsilon_1})_0\, y^{\gamma} y^{\delta} y^{\epsilon_1} + \dots. \tag{45.3}$$

In particular if these given quantities are derived as normal and metric tensors from an affine connection or fundamental metric tensor as the case may be, then the series (45.1), (45.2) and (45.3) converge. The series (45.1) determine the transformation to normal coordinates y^{α} which is given precisely by (3.2), while (45.2) and (45.3) give the components of affine connection and fundamental metric tensor in these normal coordinates. Hence if the given quantities are derivable in such a way there is one and only one affine connection or fundamental metric tensor, according to the case in question, from which they can be derived.

46. Relations between the components of the extensions of the normal tensors

Let us consider that $A^{\alpha}_{\beta\gamma\delta}(y)$ denotes a set of functions of the variables y^{α}, each of which can be expanded about the values $y^{\alpha} = 0$ in a convergent power series; we shall also assume that the functions $A^{\alpha}_{\beta\gamma\delta}$ satisfy the equations (41.7). It is desired to identify the variables y^{α} with the coordinates of a system of normal coordinates and to choose the functions $A^{\alpha}_{\beta\gamma\delta}$ in such a way, by imposing further necessary conditions, that they will be the components $A^{\alpha}_{\beta\gamma\delta}(y)$ of a normal tensor in this system of coordinates(4). This means that we wish to choose the functions $A^{\alpha}_{\beta\gamma\delta}(y)$ so that the system of equations

$$A^{\alpha}_{\beta\gamma\delta}(y) = \frac{\partial C^{\alpha}_{\beta\gamma}}{\partial y^{\delta}} - C^{\alpha}_{\beta\gamma\delta} - C^{\alpha}_{\mu\gamma} C^{\mu}_{\beta\delta} - C^{\alpha}_{\beta\mu} C^{\mu}_{\gamma\delta} \tag{46.1}$$

will have a solution given by a set of functions $C^{\alpha}_{\beta\gamma}\ (= C^{\alpha}_{\gamma\beta})$, each of which is analytic in the neighbourhood of the values $y^{\alpha} = 0$, and such that the equations

$$C^{\alpha}_{\beta\gamma}(y)\, y^{\beta} y^{\gamma} = 0 \tag{46.2}$$

are satisfied identically in the variables y^{α}. To this end we observe that the functions $A^{\alpha}_{\beta\gamma\delta}$ chosen initially so as to satisfy only equations (41.7) determine an infinite sequence of sets of constants

$$(A^{\alpha}_{\beta\gamma\delta\epsilon_1})_0;\quad (A^{\alpha}_{\beta\gamma\delta\epsilon_1\epsilon_2})_0;\quad \dots, \tag{I}$$

as explained in § 43. The constants (I) determine the formal series for the functions $C^{\alpha}_{\beta\gamma}$, namely

$$C^{\alpha}_{\beta\gamma} = (A^{\alpha}_{\beta\gamma\delta})_0\, y^{\delta} + \frac{1}{2\,!}(A^{\alpha}_{\beta\gamma\delta\epsilon_1})_0\, y^{\delta} y^{\epsilon_1} + \dots, \tag{46.3}$$

where $(A^{\alpha}_{\beta\gamma\delta})_0$ has the value $A^{\alpha}_{\beta\gamma\delta}(0)$. The series (46.3) evidently satisfy (46.2) since the quantities $(A)_0$ satisfy the complete sets of identities (41.1) and (41.2). The requirement that (46.3) shall constitute a formal solution of the differential equations (46.1) leads to other conditions on the functions

$A^{\alpha}_{\beta\gamma\delta}(y)$ than those given by (41.7). Thus by repeated differentiation of (46.1) and evaluation at $y^{\alpha}=0$, we obtain equations

$$(46.4)\qquad A^{\alpha}_{\beta\gamma\delta,\epsilon_1\ldots\epsilon_r}=\tfrac{2}{3}(A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r})_0-\tfrac{1}{3}(A^{\alpha}_{\gamma\delta\beta\epsilon_1\ldots\epsilon_r})_0-\tfrac{1}{3}(A^{\alpha}_{\delta\beta\gamma\epsilon_1\ldots\epsilon_r})_0+\star,$$

where the $\star$ represents an expression which is quadratic and homogeneous in the components of normal tensors of lower order than those which have been written down explicitly; also

$$A^{\alpha}_{\beta\gamma\delta,\epsilon_1\ldots\epsilon_r}=\left(\frac{\partial^r A^{\alpha}_{\beta\gamma\delta}(y)}{\partial y^{\epsilon_1}\ldots\partial y^{\epsilon_r}}\right)_{y^{\alpha}=0}.$$

In (46.4) the right members as well as the left are symmetric in the indices $\epsilon_1,\ldots,\epsilon_r$ arising from the differentiation; no new relations can therefore result from (46.4) as conditions of integrability. Now eliminate the above constants (I) which stand in the right members of (46.4) by a substitution of the type (43.9). This will give a system of equations of the general form

$$(46.5)\qquad A^{\alpha}_{\beta\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)=L[A^{\alpha}_{\beta\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)]+M[A^{\alpha}_{\beta\gamma\delta}(0);\ \ldots;\ A^{\alpha}_{\beta\gamma\delta,\epsilon_1\ldots\epsilon_{r-2}}(0)],$$

where, as usual, L has been used to denote a simple linear sum and M denotes a sum of homogeneous polynomials of its arguments. Equations (46.5) give conditions on the quantities $A^{\alpha}_{\beta\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)$. If the functions $A^{\alpha}_{\beta\gamma\delta}(y)$ satisfy (41.7) and are furthermore such that the quantities $A^{\alpha}_{\beta\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)$ will satisfy the sequence of equations (46.5) obtained by taking $r=1,2,\ldots$, then (46.3) will constitute a unique formal solution of the equations (46.1) and (46.2). It will be shown in § 47 that the formal solution given by (46.3) converges so that the equations (46.2) therefore give an actual characterization of the variables y^{α} as normal coordinates.

THEOREM A. *The set of functions $A^{\alpha}_{\beta\gamma\delta}(y)$, each of which is analytic in the neighbourhood of the values $y^{\alpha}=0$, will be the components of a normal tensor in a system of normal coordinates y^{α} if, and only if, the functions $A^{\alpha}_{\beta\gamma\delta}(y)$ satisfy* (41.7), *and are furthermore such that the coefficients $A^{\alpha}_{\beta\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)$ of their power series expansions about the values $y^{\alpha}=0$ satisfy the sequence of equations* (46.5).

The precise form of the equations (46.5) for $r=1$ can be obtained by eliminating the quantities $A^{\alpha}_{\beta\gamma\delta\epsilon}$ in the right members of (43.1) by means of (43.6). This gives

$$(46.6)\qquad 3A^{\alpha}_{\beta\gamma\delta,\epsilon}=3A^{\alpha}_{\beta\gamma\epsilon,\delta}+2A^{\alpha}_{\beta\epsilon\delta,\gamma}+2A^{\alpha}_{\gamma\epsilon\delta,\beta}+A^{\alpha}_{\delta\epsilon\beta,\gamma}+A^{\alpha}_{\delta\epsilon\gamma,\beta}$$

as the conditions to be satisfied by the quantities

$$A^{\alpha}_{\beta\gamma\delta,\epsilon}=A^{\alpha}_{\beta\gamma\delta,\epsilon}(0).$$

Now consider a set of constants $(g_{\alpha\beta})_0=(g_{\beta\alpha})_0$ such that the determinant $|(g_{\alpha\beta})_0|\neq 0$ and a set of functions $g_{\alpha\beta,\gamma\delta}(y)$ each of which is analytic in the neighbourhood of the values $y^{\alpha}=0$. Let us state the conditions that must be satisfied by the functions $g_{\alpha\beta,\gamma\delta}(y)$ so that they will be the components of

a metric tensor resulting from a fundamental tensor with symmetric components $\psi_{\alpha\beta}(y)$ in a system of normal coordinates y^α, such that $\psi_{\alpha\beta}(0)=(g_{\alpha\beta})_0$ at the origin of the system. In other words we wish to find conditions on the functions $g_{\alpha\beta,\gamma\delta}(y)$ so that there will exist a solution of the system of differential equations

$$(46.7)\qquad \begin{aligned} g_{\alpha\beta,\gamma\delta}(y) = {} & \frac{\partial^2\psi_{\alpha\beta}}{\partial y^\gamma \partial y^\delta} - \psi_{\sigma\beta} C^\sigma_{\alpha\gamma\delta} - \psi_{\alpha\sigma} C^\sigma_{\beta\gamma\delta} - \frac{\partial\psi_{\alpha\beta}}{\partial y^\sigma} C^\sigma_{\gamma\delta} \\ & - \frac{\partial\psi_{\sigma\beta}}{\partial y^\gamma} C^\sigma_{\alpha\delta} - \frac{\partial\psi_{\sigma\beta}}{\partial y^\delta} C^\sigma_{\alpha\gamma} - \frac{\partial\psi_{\alpha\sigma}}{\partial y^\gamma} C^\sigma_{\beta\delta} \\ & \frac{\partial\psi_{\alpha\sigma}}{\partial y^\delta} C^\sigma_{\beta\gamma} - \psi_{\sigma\tau} C^\sigma_{\alpha\gamma} C^\tau_{\beta\delta} - \psi_{\sigma\tau} C^\sigma_{\alpha\delta} C^\tau_{\beta\gamma}, \end{aligned}$$

given by a set of functions $\psi_{\alpha\beta}\,(=\psi_{\beta\alpha})$ each of which is analytic in the neighbourhood of the values $y^\alpha=0$, which are such that $\psi_{\alpha\beta}(0)=(g_{\alpha\beta})_0$, and which satisfy the equations

$$(46.8)\qquad \frac{\partial\psi_{\alpha\beta}}{\partial y^\gamma} y^\beta y^\gamma = 0$$

identically in the variables y^α. We first assume that the functions $g_{\alpha\beta,\gamma\delta}(y)$ satisfy the complete set of identities (41.14) and (41.15) as this is a necessary condition. The process described in § 45 will then determine the sets of constants

$$(\mathrm{II})\qquad (g_{\alpha\beta,\gamma\delta\epsilon_1})_0;\quad (g_{\alpha\beta,\gamma\delta\epsilon_1\epsilon_2})_0;\ \cdots$$

in such a way that the formal series for $\psi_{\alpha\beta}$, namely

$$(46.9)\qquad \psi_{\alpha\beta} = (g_{\alpha\beta})_0 + \frac{1}{2!}(g_{\alpha\beta,\gamma\delta})_0\, y^\gamma y^\delta + \frac{1}{3!}(g_{\alpha\beta,\gamma\delta\epsilon_1})_0\, y^\gamma y^\delta y^{\epsilon_1} + \ldots,$$

will satisfy (46.8); in the series (46.9) the constant $(g_{\alpha\beta,\gamma\delta})_0$ has the value $g_{\alpha\beta,\gamma\delta}(0)$. Now differentiate (46.7) repeatedly, evaluate at $y^\alpha=0$ and eliminate the quantities $(g_{\alpha\beta,\gamma\delta\epsilon_1\ldots\epsilon_r})_0$ in the right members of the resulting equations by a substitution of the type (43.15). This gives conditions analogous to (46.5), namely

$$(46.10)\qquad \begin{aligned} g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_r}(0) = {} & L[g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)] \\ & + M[(g_{\alpha\beta})_0;\ g_{\alpha\beta,\gamma\delta}(0);\ \cdots;\ g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_{r-2}}(0)], \end{aligned}$$

which must be satisfied by the quantities

$$g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_r}(0) = \left(\frac{\partial^r g_{\alpha\beta,\gamma\delta}(y)}{\partial y^{\epsilon_1}\ldots\partial y^{\epsilon_r}}\right)_{y^\alpha=0}$$

in order that the series (46.9) should constitute a formal solution of (46.7). As it will be shown in § 47 that this formal solution (46.8) converges we have the following theorem.

THEOREM B. *Given a set of constants $(g_{\alpha\beta})_0=(g_{\beta\alpha})_0$, such that the determinant $|(g_{\alpha\beta})_0|\neq 0$, and a set of functions $g_{\alpha\beta,\gamma\delta}(y)$ each of which is analytic in the neighbourhood of the values $y^\alpha=0$. There will then exist a fundamental*

tensor having components $\psi_{\alpha\beta}(y)=\psi_{\beta\alpha}(y)$ *in a system of normal coordinates* y^α *such that* (1) $\psi_{\alpha\beta}(0)=(g_{\alpha\beta})_0$ *and* (2) *the functions* $g_{\alpha\beta,\gamma\delta}(y)$ *are derivable from the* $\psi_{\alpha\beta}(y)$ *as the components of a metric tensor if, and only if, the functions* $g_{\alpha\beta,\gamma\delta}(y)$ *satisfy equations* (41.14) *and* (41.15), *and are furthermore such that the coefficients* $g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)$ *of their power series expansions about the values* $y^\alpha=0$ *satisfy the sequence of equations* (46.10).

To deduce equations corresponding to (46.6), differentiate (46.7) and evaluate at $y^\alpha=0$. This gives

$$(46.11)\qquad 6g_{\alpha\beta,\gamma\delta,\epsilon}=4g_{\alpha\beta,\gamma\delta\epsilon}+g_{\alpha\epsilon,\beta\gamma\delta}+g_{\beta\epsilon,\alpha\gamma\delta}+2g_{\gamma\delta,\alpha\beta\epsilon},$$

where $g_{\alpha\beta,\gamma\delta\epsilon}$ denotes the third derivative of $\psi_{\alpha\beta}$ with respect to y^γ, y^δ, y^ϵ evaluated at $y^\alpha=0$, and

$$g_{\alpha\beta,\gamma\delta,\epsilon}=\left(\frac{\partial g_{\alpha\beta,\gamma\delta}(y)}{\partial y^\epsilon}\right)_{y^\alpha=0}.$$

Eliminating the quantities $g_{\alpha\beta,\gamma\delta\epsilon}$ in the right members of (46.11) by (43.17) we deduce

$$(46.12)\qquad 2g_{\alpha\beta,\gamma\delta,\epsilon}=g_{\alpha\beta,\delta\epsilon,\gamma}+g_{\alpha\beta,\gamma\epsilon,\delta}+g_{\beta\epsilon,\gamma\delta,\alpha}+g_{\alpha\epsilon,\gamma\delta,\beta}.$$

It is evident in view of the above Theorem A that *equations* (43.4) *for* $s=p$ *and equations* (46.5) *for* $r=p$ *constitute a complete set of identities of the components* $A^\alpha_{\beta\gamma\delta,\epsilon_1\ldots\epsilon_p}$, and similarly it follows on account of Theorem B that *a complete set of identities for the components* $g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_p}$ *is given by equations* (43.16) *for* $s=p$ *and* (46.10) *for* $r=p$; here the components in (46.5) and (46.10) are to be regarded as associated with an arbitrary point P of the region $\mathscr{R}$, rather than with the origin of a particular system of normal coordinates as implied by the notation in these equations. In fact the more general interpretation of (46.6) and (46.12), in accordance with which the components in these equations are defined throughout the region $\mathscr{R}$, will be of importance later.

A generalization of the identities (46.5) and (46.10) corresponding to the generalizations of § 44 can obviously be made. It is evident also that we can make an extension of the above theorems to arbitrary coordinates x^ν in which the quantities Γ, occurring as the coefficients of the power series expansions of the arbitrary functions Z^ν in § 45, will enter as part of the arbitrary data of the problem.

47. Convergence proofs

A proof of the convergence of the series (46.3) and (46.9) can be made by the method involving the use of dominant functions, i.e. the *Calcul des limites* of Cauchy. Let us first prove the convergence of the series (46.3) where it is to be understood that this series is determined as above explained by a set of functions $A^\alpha_{\beta\gamma\delta}(y)$ which satisfy the conditions stated in Theorem A. It can be shown readily that the function $\mathfrak{G}^\alpha_{\beta\gamma}$ defined as a solution of the system of differential equations

$$(47.1)\qquad \frac{\partial\mathfrak{G}^\alpha_{\beta\gamma}}{\partial y^\delta}=4\mathfrak{A}^\alpha_{\beta\gamma\delta}+2\,(\mathfrak{G}^\alpha_{\mu\gamma}\;\mathfrak{Z}^\mu_{\beta\delta}+\mathfrak{G}^\alpha_{\mu\beta}\;\mathfrak{G}^\mu_{\gamma\delta}),$$

such that $\mathfrak{C}^{\alpha}_{\beta\gamma}=0$ when $y^1=\ldots=y^n=0$ is a dominant function for the component $C^{\alpha}_{\beta\gamma}$ given by (46.3); in (47.1) the function $\mathfrak{A}^{\alpha}_{\beta\gamma\delta}$ is a dominant function for $A^{\alpha}_{\beta\gamma\delta}(y)$. Equations (47.1) can in fact be reduced to a very simple form by putting

$$\mathfrak{C}^{\alpha}_{\beta\gamma}=\Phi, \quad \mathfrak{A}^{\alpha}_{\beta\gamma\delta}=F,$$

where F, defined by the equation

$$F=\frac{M}{1-(y^1+\ldots+y^n)/\rho} \tag{47.2}$$

for suitable positive constants M and ρ, is a dominant function for any of the functions $A^{\alpha}_{\beta\gamma\delta}$. With the above substitutions (47.1) becomes

$$\frac{d\Phi}{dy}=4F+4n\Phi^2, \tag{47.3}$$

where $y=y^1+\ldots+y^n$. The existence of a solution Φ of the equation (47.3) which is analytic in the neighbourhood of $y=0$, and such that $\Phi=0$ for $y=0$, results from the well-known theorem of differential equations. To show that the function Φ satisfying these initial conditions dominates the series (46.3) for the affine connection $C^{\alpha}_{\beta\gamma}$ let us differentiate (47.1) repeatedly and evaluate the resulting equations at the point $y^{\alpha}=0$. We thus obtain expressions of the general form

$$\Phi^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}=4F^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}+2\left[\frac{\partial^r(\mathfrak{C}^{\alpha}_{\mu\gamma}\mathfrak{C}^{\mu}_{\beta\delta}+\mathfrak{C}^{\alpha}_{\mu\delta}\mathfrak{C}^{\mu}_{\beta\gamma})}{\partial y^{\epsilon_1}\ldots\partial y^{\epsilon_r}}\right]_{y^{\alpha}} \tag{47.4}$$

where $\Phi^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}$ and $F^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}$ denote the derivatives of $\partial\mathfrak{C}^{\alpha}_{\beta\gamma}/\partial y^{\delta}$ and $\mathfrak{A}^{\alpha}_{\beta\gamma\delta}$ with respect to $y^{\epsilon_1}, \ldots, y^{\epsilon_r}$ evaluated at $y^{\alpha}=0$. Assuming that

$$\Phi^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_s}>|(A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_s})_0| \tag{47.5}$$

for $s<r\,(\geqq 2)$, we can show that for $s=r$ these inequalities are likewise satisfied. Compare equations (47.4) with equations

$$A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}=A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}+A^{\alpha}_{\beta\gamma\delta,\,\epsilon_1\ldots\epsilon_r}-A^{\alpha}_{\beta\delta\gamma,\,\epsilon_1\ldots\epsilon_r}+\star \tag{47.6}$$

(see §43), in which $A^{\alpha}_{\beta\gamma\delta}$ in the $\star$ term has the value $A^{\alpha}_{\beta\gamma\delta}(0)$, and the quantities $A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}$ and $A^{\alpha}_{\beta\gamma\delta,\,\epsilon_1\ldots\epsilon_r}$ are considered to represent the constants $(A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r})_0$ and $A^{\alpha}_{\beta\gamma\delta,\,\epsilon_1\ldots\epsilon_r}(0)$ for any value of $r\,(\geqq 1)$. This comparison shows that

$$2\left[\frac{\partial^r(\mathfrak{C}^{\alpha}_{\mu\gamma}\mathfrak{C}^{\mu}_{\beta\delta}+\mathfrak{C}^{\alpha}_{\mu\delta}\mathfrak{C}^{\mu}_{\beta\gamma})}{\partial y^{\epsilon_1}\ldots\partial y^{\epsilon_r}}\right]_{y^{\alpha}=0}$$

is greater than twice the absolute value of the $\star$ terms in (47.6) in virtue of (47.5) for $s<r$; also $4F^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}$ is greater than twice the absolute value of the difference $A^{\alpha}_{\beta\gamma\delta,\,\epsilon_1\ldots\epsilon_r}-A^{\alpha}_{\beta\delta\gamma,\,\epsilon_1\ldots\epsilon_r}$ in (47.6) because of the dominating property of the function F. Hence

$$\Phi^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}>2\,|(A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r})_0-(A^{\alpha}_{\beta\delta\gamma\epsilon_1\ldots\epsilon_r})_0|. \tag{47.7}$$

Since the $\Phi^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}$ are symmetric with respect to all indices, we also have

$$\Phi^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}>2\,|(A^{\alpha}_{\beta\delta\gamma\epsilon_1\ldots\epsilon_r})_0-(A^{\alpha}_{\delta\epsilon_1\beta\gamma\ldots\epsilon_r})_0|. \tag{47.8}$$

Adding (47.7) and (47.8) we obtain

$$\Phi^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}>|(A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r})_0-(A^{\alpha}_{\delta\epsilon_1\beta\gamma\ldots\epsilon_r})_0|.$$

Hence $\Phi^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}$ is greater than the absolute value of the difference of $(A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r})_0$ and $(A^{\alpha}_{\mu\nu\sigma\tau_1\ldots\tau_r})_0$, where $\mu, \nu, \sigma, \tau_1, \ldots, \tau_r$ is any permutation of $\beta, \gamma, \delta, \epsilon_1, \ldots, \epsilon_r$; we can therefore write

$$\Phi^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}>|(A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r})_0-(A^{\alpha}_{\mu\nu\sigma\tau_1\ldots\tau_r})_0|. \tag{47.9}$$

Now form all permutations of the indices $\mu, \nu, \sigma, \tau_1, \ldots, \tau_r$ and add together the $\mathfrak{N}$ corresponding inequalities (47.9), obtaining

$$\mathfrak{N}\Phi^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r}>|\mathfrak{N}(A^{\alpha}_{\beta\gamma\delta\epsilon_1\ldots\epsilon_r})_0-\Sigma(A^{\alpha}_{\mu\nu\sigma\tau_1\ldots\tau_r})_0|.$$

But the sum in the right member of this inequality vanishes by (41.2). Hence we obtain the inequality (47.5) for $s=r(\geqq 2)$ which we wished to prove. The fact that (47.5) holds for $s=1$ is proved directly by observing that for this case the $\star$ terms in (47.6) and the second set of terms in (47.4) vanish; we are thus led to the inequality (47.7) for $r=1$ and hence to (47.5) for $s=1$. The function $\mathfrak{C}^{\alpha}_{\beta\gamma}$ therefore dominates the series (46.3) for the affine connection $C^{\alpha}_{\beta\gamma}$ with the result that (46.3) must converge.

To prove the convergence of the series (46.9) we set up the system of differential equations

$$\partial^2 \mathfrak{H}_{\alpha\beta}/\partial y^{\gamma}\,\partial y^{\delta}=4\mathfrak{G}_{\alpha\beta\gamma\delta}+4\Sigma\,[(\partial\mathfrak{H}_{\sigma\alpha}/\partial y^{\beta})\,\mathfrak{N}^{\sigma}_{\gamma\delta}+\mathfrak{H}_{\sigma\tau}\,\mathfrak{N}^{\sigma}_{\alpha\beta}\,\mathfrak{N}^{\tau}_{\gamma\delta}], \tag{47.10}$$

where $\mathfrak{G}_{\alpha\beta\gamma\delta}$ is a dominant function for $g_{\alpha\beta,\,\gamma\delta}(y)$ and the summation Σ denotes the sum of all terms that can be formed from the term

$$(\partial\mathfrak{H}_{\sigma\alpha}/\partial y^{\beta})\,\mathfrak{N}^{\sigma}_{\gamma\delta}+\mathfrak{H}_{\sigma\tau}\,\mathfrak{N}^{\sigma}_{\alpha\beta}\,\mathfrak{N}^{\tau}_{\gamma\delta} \tag{47.11}$$

by taking all permutations of the indices $\alpha, \beta, \gamma, \delta$. Also the expression $\mathfrak{N}^{\alpha}_{\beta\gamma}$ in the term (47.11) is defined by

$$\mathfrak{N}^{\alpha}_{\beta\gamma}=\mathfrak{L}^{\alpha\sigma}\,[(\partial\mathfrak{H}_{\sigma\beta}/\partial y^{\gamma})+(\partial\mathfrak{H}_{\sigma\gamma}/\partial y^{\beta})+(\partial\mathfrak{H}_{\beta\gamma}/\partial y^{\sigma})],$$

where $\mathfrak{L}^{\alpha\sigma}$ is a dominant function for $\mathfrak{H}^{\alpha\sigma}$. Now replace $\mathfrak{H}_{\alpha\beta}$ by Ψ and take

$$\Psi_0>|\,(g_{\alpha\beta})_0\,|\quad\text{(absolute value);}$$

also take

$$\mathfrak{G}_{\alpha\beta\gamma\delta}=\frac{M}{1-(y/\rho)},\quad \mathfrak{L}^{\alpha\sigma}=\frac{M'}{1-(\Psi-\Psi_0)/\rho'}$$

where $y=y^1+\ldots+y^n$ and M, M', ρ, ρ' are suitable positive constants. The equations (47.10) will then reduce to a single differential equation as in the case previously considered, when Ψ is assumed to be a function of the single variable y alone. This differential equation possesses a unique solution $\Psi(y)$ expandable in a convergent power series about $y=0$ such that $\Psi(0)=\Psi_0$ and $d\Psi(y)/dy=0$ for $y=0$. The function $\Psi(y)$ so determined dominates the series (46.9). To show this we assume that

$$\Psi_{\alpha\beta\gamma\delta\epsilon_1\ldots\epsilon_s}>|\,(g_{\alpha\beta,\,\gamma\delta\epsilon_1\ldots\epsilon_s})_0\,| \tag{47.12}$$

for $s<r(\geqq 2)$, where the quantity $\Psi_{\alpha\beta\gamma\delta\epsilon_1\ldots\epsilon_s}$ is equal to the derivative of $\mathfrak{H}_{\alpha\beta}$ with respect to $y^{\gamma}, \ldots, y^{\epsilon_s}$ evaluated at $y^{\alpha}=0$. We shall show that (47.12) is also true for $s=r$ which will prove the dominating property of the function $\Psi(y)$. Due to the fact that $\mathfrak{H}_{\alpha\beta}$ is to be taken equal to the function $\Psi(y)$ it follows that $\Psi_{\alpha\beta\gamma\delta\epsilon_1\ldots\epsilon_s}$ is symmetric with respect to all its indices. To prove (47.12) for $s=r$ we differentiate (47.10) repeatedly r times, then evaluate at $y^{\alpha}=0$ and compare the resulting equations with the equations

$$\begin{aligned} &g_{\alpha\beta,\,\gamma\delta\epsilon_1\ldots\epsilon_r}-g_{\alpha\gamma,\,\beta\delta\epsilon_1\ldots\epsilon_r}+g_{\delta\gamma,\,\beta\alpha\epsilon_1\ldots\epsilon_r}-g_{\delta\beta,\,\gamma\alpha\epsilon_1\ldots\epsilon_r} \\ &\qquad=g_{\alpha\beta,\,\gamma\delta,\,\epsilon_1\ldots\epsilon_r}-g_{\alpha\gamma,\,\beta\delta,\,\epsilon_1\ldots\epsilon_r}+g_{\delta\gamma,\,\beta\alpha,\,\epsilon_1\ldots\epsilon_r}-g_{\delta\beta,\,\gamma\alpha,\,\epsilon_1\ldots\epsilon_r}+\star \end{aligned} \tag{47.13}$$

[equations (43.13)], in which the $g_{\alpha\beta,\,\gamma\delta}$ appearing in the $\star$ terms has the value $g_{\alpha\beta,\,\gamma\delta}(0)$ and the quantities $g_{\alpha\beta,\,\gamma\delta\epsilon_1\ldots\epsilon_r}$ and $g_{\alpha\beta,\,\gamma\delta,\,\epsilon_1\ldots\epsilon_r}$ have the values $(g_{\alpha\beta,\,\gamma\delta\epsilon_1\ldots\epsilon_r})_0$ and $g_{\alpha\beta,\,\gamma\delta,\,\epsilon_1\ldots\epsilon_r}(0)$, respectively, for any value of $r(\geqq 1)$. It is clear that

$$\Psi_{\alpha\beta\gamma\delta\epsilon_1\ldots\epsilon_r}>|\,(g_{\alpha\beta,\,\gamma\delta\epsilon_1\ldots\epsilon_r})_0-(g_{\alpha\gamma,\,\beta\delta\epsilon_1\ldots\epsilon_r})_0+(g_{\delta\gamma,\,\beta\alpha\epsilon_1\ldots\epsilon_r})_0-(g_{\delta\beta,\,\gamma\alpha\epsilon_1\ldots\epsilon_r})_0\,|\,. \tag{47.14}$$

We now introduce the operations P and P' defined in § 43. It will be recalled that the operation P on a term such as $g_{\alpha\beta,\,\gamma\delta\epsilon_1\ldots\epsilon_r}$ effects the summation of all terms obtainable from $g_{\alpha\beta,\,\gamma\delta\epsilon_1\ldots\epsilon_r}$ by permuting the indices $\delta, \epsilon_1, \ldots, \epsilon_r$ cyclically while the remaining indices α, β, γ are held fixed; similarly the operation P' on the term $g_{\alpha\beta,\,\gamma\delta\epsilon_1\ldots\epsilon_r}$ effects the summation of the terms obtainable from $g_{\alpha\beta,\,\gamma\delta\epsilon_1\ldots\epsilon_r}$ by taking all permutations of the indices $\gamma, \delta, \epsilon_1, \ldots, \epsilon_r$. First operating on both members of (47.14) by P, we have

$$(r+1)\,\Psi_{\alpha\beta\gamma\delta\epsilon_1\ldots\epsilon_r}>(r+2)\,|\,(g_{\alpha\beta,\,\gamma\delta\epsilon_1\ldots\epsilon_r})_0-(g_{\alpha\gamma,\,\beta\delta\epsilon_1\ldots\epsilon_r})_0\,|\,.$$

We next operate on both members of these latter inequalities by P' which gives

$$(r+1)\,\Psi^*_{\alpha\beta\gamma\delta\epsilon_1\ldots\epsilon_r} > (r+3)\,|(g_{\alpha\beta,\gamma\delta\epsilon_1\ldots\epsilon_r})_0|.$$

This proves (47.12) for $s=r\,(\geqq 2)$. If $s=1$ we obtain the inequality (47.14) for $r=1$ since the $\star$ terms in (47.10) and (47.13) then vanish; operating on (47.14) for $r=1$ as above, we obtain (47.12) for $s=1$. The function $\Psi^*(y)$ is therefore a dominant function for the series (46.9), and therefore the series (46.9) converges within a sufficiently small neighbourhood of the values $y^\alpha=0$.

48. Relations between the components of certain invariants of the space of distant parallelism

The identities and theorems of § 43 to § 46 are likewise true in the space of distant parallelism since this space bears an affine connection Δ. It is evident, however, that each of these identities and theorems admits an analogue in the space of distant parallelism in terms of the scalar derivatives $h^i_{j,k\ldots m}$ defined in § 34 with the aid of absolute coordinates. We shall limit ourselves here to the derivation of a few of these identities which are of considerable importance in our later work (5).

We have

$$h^i_{j,k,l} = \frac{\partial h^i_{j,k}}{\partial x^\alpha} h^\alpha_l; \tag{48.1}$$

hence

$$h^i_{j,k,l} = \tfrac{1}{2}[h^i_{j,kl} - h^i_{k,jl}] + h^i_{m,j} h^m_{k,l} + h^i_{m,k} h^m_{l,j}, \tag{48.2}$$

when use is made of (35.14) and (42.3). Interchanging k, l in (48.2) and adding, we obtain a set of identities which can be reduced to the form

$$h^i_{j,kl} = \tfrac{2}{3}[h^i_{j,k,l} + h^i_{j,l,k} + h^i_{m,k} h^m_{j,l} + h^i_{m,l} h^m_{j,k}]. \tag{48.3}$$

We observe that (48.3) constitutes an inverse form of (48.2) in that (48.2) expresses the $h^i_{j,k,l}$ in terms of the $h^i_{j,kl}$, while (48.3) gives the $h^i_{j,kl}$ in terms of the $h^i_{j,k,l}$.

The invariants $h^i_{j,k,l}$ satisfy a set of identities

$$h^i_{j,k,l} + h^i_{k,l,j} + h^i_{l,j,k} = 2\,[h^i_{m,j} h^m_{k,l} + h^i_{m,k} h^m_{l,j} + h^i_{m,l} h^m_{j,k}], \tag{48.4}$$

which is easily deduced from (48.2).

Let us now transform the expression for $h^i_{j,k}$ given by (35.13) to a system of absolute normal coordinates, differentiate repeatedly, and evaluate at the origin of the system. We thereby obtain a system of equations of the form

$$h^i_{j,k,l_1\ldots l_r} = \tfrac{1}{2}[h^i_{j,kl_1\ldots l_r} - h^i_{k,jl_1\ldots l_r}] + \star, \tag{48.5}$$

where the $\star$ is used to denote scalar derivatives of lower order than those which have been written down explicitly. Now let P denote the operations of holding j fixed, permuting the indices $k, l_1, \ldots, l_r$ cyclically, and adding the resulting terms. Then

$$2P(h^i_{j,k,l_1\ldots l_r}) = P(h^i_{j,kl_1\ldots l_r}) - P(h^i_{k,jl_1\ldots l_r}) + \star$$

from (48.5); whence using (42.1) and (42.2) we obtain

$$(48.6) \qquad (r+2)\, h^i_{j,\,kl_1\ldots l_r} = 2P\,(h^i_{j,\,k,\,l_1\ldots l_r}) + \text{☆}.$$

Using (48.6) as a recurrence formula, we have

$$(48.7) \qquad (r+2)\, h^i_{j,\,kl_1\ldots l_r} = 2P\,(h^i_{j,\,k,\,l_1\ldots l_r}) + Q\,[h^i_{j,\,k};\ \ldots;\ h^i_{j,\,k,\,l_1\ldots l_{r-1}}],$$

where the expression Q is a polynomial in the quantities indicated. In particular for $r=1$, the identities (48.7) become the identities (48.3) already found.

49. Determination of the components of the affine normal tensors in terms of the components of the curvature tensor and its covariant derivatives

Let us consider an affine space bearing a symmetric affine connection with components $\Gamma^\alpha_{\beta\gamma}$; the extension of the following results to the case where the affine connection is not symmetric can readily be made. It is then an immediate consequence of the replacement theorem of § 39 that the components of the affine curvature tensor or any of its covariant derivatives can be expressed in terms of the components of affine normal tensors A. To derive these equations, we observe that

$$(49.1) \qquad B^\alpha_{\beta\gamma\delta,\,\ldots,\,\sigma} = \frac{\partial^r \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta \ldots \partial x^\sigma} - \frac{\partial^r \Gamma^\alpha_{\beta\delta}}{\partial x^\gamma \ldots \partial x^\sigma} + \text{☆},$$

where the ☆ indicates a polynomial in the $\Gamma^\alpha_{\beta\gamma}$ and their derivatives of lower order than those which have been written down explicitly (6). Evaluation of (49.1) at the origin of a system of affine normal coordinates gives

$$(49.2) \qquad B^\alpha_{\beta\gamma\delta,\,\ldots,\,\sigma} = A^\alpha_{\beta\gamma\delta\ldots\sigma} - A^\alpha_{\beta\delta\gamma\ldots\sigma} + \text{☆},$$

where now the ☆ denotes a polynomial in the components of normal tensors A which involve fewer subscripts than are contained in the set $\beta, \gamma, \delta, \ldots, \sigma$.

We can obtain identities which express the components of the normal tensors A in terms of the components of the curvature tensor and its covariant derivatives by combining (49.2) with (41.1) and (41.2) in a rather obvious manner. For example, the component $A^\alpha_{\beta\delta\gamma\ldots\sigma}$ can be eliminated from (41.2) by the substitution (49.2) which has the effect of replacing the component $A^\alpha_{\beta\delta\gamma\ldots\sigma}$ in (41.2) directly by the component $B^\alpha_{\beta\gamma\delta,\,\ldots,\,\sigma}$ plus the expression in the other components which occur in (49.2); a similar substitution is to be made on any component in (41.2) which contains either β or γ among its first two subscripts. In case neither β nor γ appears among the first two subscripts of the component in (41.2) two substitutions of the above type are required. In this way we obtain a set of identities of the form

$$A^\alpha_{\beta\gamma\delta\ldots\sigma} = L\,(B^\alpha_{\beta\gamma\delta,\,\ldots,\,\sigma}) + \text{☆},$$

where L denotes a linear sum of the components indicated and the ☆ has

the same significance as in (49.2). Use of the above equations as recurrence equations gives the required identities

$$A^{\alpha}_{\beta\gamma\delta\ldots\sigma} = L(B^{\alpha}_{\beta\gamma\delta,\ \ldots,\ \sigma}) + M, \tag{49.3}$$

where M denotes a polynomial expression in components of the curvature tensor and its covariant derivatives involving fewer subscripts than those in the set $\beta, \gamma, \delta, \ldots, \sigma$.

Particular cases of (49.2) are

$$B^{\alpha}_{\beta\gamma\delta} = A^{\alpha}_{\beta\gamma\delta} - A^{\alpha}_{\beta\delta\gamma}, \tag{49.4}$$

$$B^{\alpha}_{\beta\gamma\delta,\,\epsilon} = A^{\alpha}_{\beta\gamma\delta\epsilon} - A^{\alpha}_{\beta\delta\gamma\epsilon}; \tag{49.5}$$

for these cases the ☆ terms in (49.2) vanish. The corresponding particular cases of (49.3) are

$$3A^{\alpha}_{\beta\gamma\delta} = B^{\alpha}_{\beta\gamma\delta} + B^{\alpha}_{\gamma\beta\delta}, \tag{49.6}$$

$$6A^{\alpha}_{\beta\gamma\delta\epsilon} = 2B^{\alpha}_{\beta\gamma\delta,\,\epsilon} + B^{\alpha}_{\beta\gamma\epsilon,\,\delta} + B^{\alpha}_{\gamma\beta\delta,\,\epsilon} + B^{\alpha}_{\gamma\beta\epsilon,\,\delta} + B^{\alpha}_{\delta\beta\epsilon,\,\gamma}, \tag{49.7}$$

as can easily be obtained from (49.4) and (49.5) by the above general procedure.

Use of (49.2) and (49.3) enables us to determine the complete set of identities of the components of the curvature tensor or any of its covariant derivatives. Thus the identities

$$B^{\alpha}_{\beta\gamma\delta} = -B^{\alpha}_{\beta\delta\gamma}, \quad B^{\alpha}_{\beta\gamma\delta} + B^{\alpha}_{\gamma\delta\beta} + B^{\alpha}_{\delta\beta\gamma} = 0 \tag{49.8}$$

are easily seen to be true on account of (49.4) and (41.7). Conversely the equations (49.6) and (49.8) yield the equations (49.4) and (41.7). Hence we see that any set of identities in the components of the curvature tensor $B^{\alpha}_{\beta\gamma\delta}$ can be satisfied by a set of numbers $B^{\alpha}_{\beta\gamma\delta}$ which are subject only to the algebraic conditions (49.8). In other words, *the identities* (49.8) *constitute a complete set of identities of the components* $B^{\alpha}_{\beta\gamma\delta}$ *of the curvature tensor of an affine space with symmetric affine connection.*

In a similar manner we can calculate for the case of a metric space the complete set of identities satisfied by the components $B_{\alpha\beta\gamma\delta}$ of the covariant form of the curvature tensor; these components $B_{\alpha\beta\gamma\delta}$ are defined by (12.20). We can easily deduce the relations

$$B_{\alpha\beta\gamma\delta} = g_{\alpha\gamma,\,\beta\delta} - g_{\beta\gamma,\,\alpha\delta}. \tag{49.9}$$

It then follows from (49.9) and the identities (41.14) and (41.15) that

$$3g_{\alpha\beta,\,\gamma\delta} = B_{\alpha\gamma\beta\delta} + B_{\beta\gamma\alpha\delta}, \tag{49.10}$$

$$B_{\alpha\beta\gamma\delta} = -B_{\beta\alpha\gamma\delta} = -B_{\alpha\beta\delta\gamma}, \quad B_{\alpha\beta\gamma\delta} + B_{\alpha\gamma\delta\beta} + B_{\alpha\delta\beta\gamma} = 0. \tag{49.11}$$

Conversely we can obtain the identities (41.14), (41.15) and (49.9) from the identities (49.10) and (49.11). Hence it follows that *the identities* (49.11) *constitute a complete set of identities of the components* $B_{\alpha\beta\gamma\delta}$ *of the covariant form of the curvature tensor of a metric space.*

Let us now consider the problem of obtaining the complete set of identities of the components $B^{\alpha}_{\beta\gamma\delta,\epsilon}$ appearing in (49.5). We observe first that we must have the identities in these components which are obtained by covariant differentiation of (49.8), namely

$$(49.12) \qquad B^{\alpha}_{\beta\gamma\delta,\epsilon} = -B^{\alpha}_{\beta\delta\gamma,\epsilon}, \quad B^{\alpha}_{\beta\gamma\delta,\epsilon} + B^{\alpha}_{\gamma\delta\beta,\epsilon} + B^{\alpha}_{\delta\beta\gamma,\epsilon} = 0.$$

Now interchange the indices δ, ϵ in (49.7) and subtract, making use of (49.12); this gives*

$$(49.13) \qquad B^{\alpha}_{\beta\gamma\delta,\epsilon} + B^{\alpha}_{\beta\delta\epsilon,\gamma} + B^{\alpha}_{\beta\epsilon\gamma,\delta} = 0.$$

Conversely from (49.7), (49.12) and (49.13) we can deduce (49.5) and the identities (41.8) and (41.9). Hence, *the identities* (49.12) *and* (49.13) *constitute a complete set of identities of the components* $B^{\alpha}_{\beta\gamma\delta,\epsilon}$ *of the covariant derivative of the curvature tensor of an affine space with symmetric affine connection.* It is evident that a continuation of this process will lead to the determination of the complete set of identities of the components of any covariant derivative of the curvature tensor.

By summing on the indices α and β in (49.13) and (49.8) we obtain

$$B^{\alpha}_{\alpha\gamma\delta,\epsilon} + B^{\alpha}_{\alpha\delta\epsilon,\gamma} + B^{\alpha}_{\alpha\epsilon\gamma,\delta} = 0,$$

and

$$B^{\alpha}_{\alpha\gamma\delta} = B_{\delta\gamma} - B_{\gamma\delta},$$

respectively, where $B_{\beta\gamma}$ are the components of the contracted curvature tensor defined in § 52. Hence we obtain the identities

$$(49.14) \qquad B_{\delta\gamma,\epsilon} - B_{\gamma\delta,\epsilon} + B_{\epsilon\delta,\gamma} - B_{\delta\epsilon,\gamma} + B_{\gamma\epsilon,\delta} - B_{\epsilon\gamma,\delta} = 0.$$

50. Curvature. Theorem of Schur

The identities (49.11) and (49.13) furnish the basis of the proof of a theorem of Schur concerning the curvature of a metric space. To define the curvature at a point P of the region $\mathscr{R}$ covered by the coordinate system in a metric space $\mathscr{T}$, we associate with P two independent vectors λ_1 and λ_2 having components λ^{α}_{1} and λ^{α}_{2}, respectively. Then any vector λ at P linearly dependent on λ_1 and λ_2 will have components given by

$$(50.1) \qquad \lambda^{\alpha} = \sum_{i=1}^{2} a^i \lambda^{\alpha}_{i},$$

where a^i are constants, and conversely. We now consider the *geodesic surface* S formed by the totality of geodesics issuing from P in directions λ^{α} given by (50.1). Assuming at the outset that the space $\mathscr{T}$ is Riemannian (see § 5), the Gaussian curvature of the surface S at P is then called *the curvature of the space* $\mathscr{T}$ *at the point* P *for the orientation determined by the*

* This identity, usually called the *Bianchi identity*, was communicated by Ricci to E. Padova who published it in the paper "Sulle deformazioni infinitesimi", *Atti dei Lincei, Rend.* (4), 5[1] (1889), pp. 174–8. It was later proven independently by L. Bianchi, "Sui simboli a quattro indici e sulla curvatura di Riemann", *Atti dei Lincei, Rend.* (5), 11[2] (1902), pp. 3–7.

vectors λ_1 *and* λ_2. As so defined curvature is an intrinsic property independent of the coordinate system (7).

Throughout this section it will be assumed that Latin indices have values 1, 2 and Greek indices values 1, ..., n where $n \geqq 3$; the summation convention will then be applied with regard to these ranges of indices.

To derive an explicit expression for the curvature, we introduce normal coordinates y^α with origin at the above point P. Then the equations of the geodesic surface S can be written

(50.2) $$y^\alpha = \lambda_i^\alpha u^i,$$

where the u^i are parameters which can be interpreted as coordinates of the surface S. The element of distance on the surface S is then given by

$$d\sigma^2 = b_{ik} du^i du^k,$$

where

(50.3) $$b_{ik} = \psi_{\alpha\beta} \frac{\partial y^\alpha}{\partial u^i} \frac{\partial y^\beta}{\partial u^k} = \psi_{\alpha\beta} \lambda_i^\alpha \lambda_k^\beta;$$

here the $\psi_{\alpha\beta}$ represent of course the components of the fundamental metric tensor in the system of normal coordinates. Since the space $\mathscr{T}$ is Riemannian the determinant $|b_{ik}|$ will be different from zero at the point P.* Let us now put

(50.4) $$C_{\alpha\beta\gamma} = \psi_{\alpha\nu} C^\nu_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial \psi_{\alpha\beta}}{\partial y^\gamma} + \frac{\partial \psi_{\alpha\gamma}}{\partial y^\beta} - \frac{\partial \psi_{\beta\gamma}}{\partial y^\alpha}\right),$$

and let us also denote by Γ_{ikl} the corresponding quantities with reference to the surface S. Then from (50.3) we have

(50.5) $$\Gamma_{ikl} = C_{\alpha\beta\gamma} \lambda_i^\alpha \lambda_k^\beta \lambda_l^\gamma.$$

In a similar manner let us denote the components of the curvature tensors

* If θ denotes the angle determined by the two vectors λ_1 and λ_2 at P, then

$$\cos\theta = \frac{g_{\alpha\beta}\lambda_1^\alpha \lambda_2^\beta}{\sqrt{g_{\alpha\beta}\lambda_1^\alpha\lambda_1^\beta}\sqrt{g_{\alpha\beta}\lambda_2^\alpha\lambda_2^\beta}},$$

where the quantities $g_{\alpha\beta}$ are evaluated at P. Using the relation $\sin^2\theta = 1 - \cos^2\theta$, we find

$$\sin^2\theta = \frac{(g_{\gamma\alpha}g_{\beta\delta} - g_{\alpha\delta}\,g_{\beta\gamma})\,\lambda_1^\alpha\,\lambda_2^\beta\,\lambda_1^\gamma\,\lambda_2^\delta}{g_{\alpha\gamma}g_{\beta\delta}\,\lambda_1^\alpha\,\lambda_2^\beta\lambda_1^\gamma\,\lambda_2^\delta} = |\,b_{ik}\,|/g_{\alpha\gamma}g_{\beta\delta}\,\lambda_1^\alpha\,\lambda_2^\beta\,\lambda_1^\gamma\,\lambda_2^\delta.$$

Now the denominator in this expression is equal to the product of the square of the lengths of the two vectors λ_1 and λ_2 and is therefore different from zero since the space is assumed to be Riemannian; also $\sin^2\theta \neq 0$ since the vectors λ_1 and λ_2 are independent. Hence it follows that $|\,b_{ik}\,| \neq 0$ at the point P.

If, however, we have a metric space for which the fundamental quadratic form is indefinite, the determinant $|\,b_{ik}\,|$ may have the value zero at P. For example, suppose that $n = 3$ and that the matrix of the quantities $g_{\alpha\beta}$ is given by

$$\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix}$$

at P. Take $\lambda_i^\alpha = \delta_i^\alpha$. Then $b_{ik} = g_{ik}$ where $i, k = 1, 2$ and we see immediately from the above matrix that $|\,b_{ik}\,| = 0$ at P.

(see § 12) by B_{iklm} and $B_{\alpha\beta\gamma\delta}$ with reference to the surface S and the Riemann space $\mathscr{T}$, respectively. From the formula (12.20) we then have

$$B_{iklm} = b_{ij} B^j_{klm} = b_{ij} \left[\frac{\partial \Gamma^j_{kl}}{\partial u^m} - \frac{\partial \Gamma^j_{km}}{\partial u^l} + \Gamma^j_{pm} \Gamma^p_{kl} - \Gamma^j_{pl} \Gamma^p_{km} \right]$$

$$= \frac{\partial \Gamma_{ikl}}{\partial u^m} - \frac{\partial \Gamma_{ikm}}{\partial u^l} + \Gamma_{ipm} \Gamma^p_{kl} - \Gamma_{ipl} \Gamma^p_{km} - \frac{\partial b_{ij}}{\partial u^m} \Gamma^j_{kl} + \frac{\partial b_{ij}}{\partial u^l} \Gamma^j_{km}.$$

Evaluating both members of these equations at the origin of normal coordinates, we obtain

$$B_{iklm}(0) = \left(\frac{\partial \Gamma_{ikl}}{\partial u^m} \right)_0 - \left(\frac{\partial \Gamma_{ikm}}{\partial u^l} \right)_0$$

$$= \tfrac{1}{2} [(g_{\alpha\beta,\gamma\delta} + g_{\alpha\gamma,\beta\delta} - g_{\beta\gamma,\alpha\delta}) - (g_{\alpha\beta,\delta\gamma} + g_{\alpha\delta,\beta\gamma} - g_{\beta\delta,\alpha\gamma})] \lambda^\alpha_i \lambda^\beta_k \lambda^\gamma_l \lambda^\delta_m,$$

when use is made of equations (50.2), (50.3) and (50.4). Using (49.9) and the identities (41.16), these latter equations take the form

$$B_{iklm} = B_{\alpha\beta\gamma\delta} \lambda^\alpha_i \lambda^\beta_k \lambda^\gamma_l \lambda^\delta_m. \tag{50.6}$$

If we make an arbitrary analytic transformation $u^i \to \bar{u}^i$ of the coordinates of the surface S, we have

$$| \bar{b}_{ik} | = | b_{ik} | (u\bar{u})^2,$$

where $(u\bar{u})$ denotes the functional determinant of the transformation. Now the components B_{iklm} are either equal to zero or differ from the component B_{1212} at most in sign on account of the identities (49.11). In consequence of this fact we can easily deduce from the equations of transformation of the components B_{iklm}, resulting from the above transformation $u^i \to \bar{u}^i$, that

$$\bar{B}_{1212} = B_{1212} (u\bar{u})^2.$$

Hence the quantity

$$K = -\frac{B_{1212}}{| b_{ik} |} = -\frac{B_{1212}}{b_{11} b_{22} - b_{12} b_{21}} \tag{50.7}$$

is a differential invariant of the surface S and in fact is equal to the Gaussian curvature of this surface.*

From (50.3) and (50.6) we now obtain

$$K = \frac{-B_{\alpha\beta\gamma\delta} \lambda^\alpha_1 \lambda^\beta_2 \lambda^\gamma_1 \lambda^\delta_2}{(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) \lambda^\alpha_1 \lambda^\beta_2 \lambda^\gamma_1 \lambda^\delta_2} \tag{50.8}$$

as the expression for the curvature of a Riemann space ($n \geqq 3$) at a point P for the orientation determined by the vectors λ_1 and λ_2. As an extension of this result we define the value of K given by (50.8) to be the curvature in the case of a metric space ($n \geqq 3$) for which the fundamental quadratic differential form may be indefinite.

* From (50.7) it follows that

$$K b_{11} = -B^2_{121}, \quad K b_{12} = K b_{21} = B^2_{212}, \quad K b_{22} = -B^1_{212}.$$

Hence K is the Gaussian curvature of the surface S. See L. P. Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces* (Ginn and Co. 1909), p. 155.

It is evident from (50.8) that the necessary and sufficient condition that the curvature at every point P of the space be independent of the orientation is that

$$B_{\alpha\beta\gamma\delta} = -\mu(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}), \tag{50.9}$$

where μ is at most a function of the coordinates.

In a metric space we can write the identities (49.13) in the completely covariant form

$$B_{\alpha\beta\gamma\delta,\epsilon} + B_{\alpha\beta\delta\epsilon,\gamma} + B_{\alpha\beta\epsilon\gamma,\delta} = 0. \tag{50.10}$$

Assuming now that the curvature K at any point of the space is independent of the orientation, we have

$$B_{\alpha\beta\gamma\delta,\epsilon} = -\frac{\partial\mu}{\partial x^\epsilon}[g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}] \tag{50.11}$$

by covariant differentiation of (50.9). Substituting (50.11) into (50.10), we then obtain

$$\frac{\partial\mu}{\partial x^\gamma}[g_{\alpha\delta}g_{\beta\epsilon} - g_{\alpha\epsilon}g_{\beta\delta}] + \frac{\partial\mu}{\partial x^\delta}[g_{\alpha\epsilon}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\epsilon}] + \frac{\partial\mu}{\partial x^\epsilon}[g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}] = 0.$$

Multiplying these equations by $g^{\alpha\gamma}g^{\beta\delta}$ and summing on repeated indices give

$$(n-1)(n-2)\frac{\partial\mu}{\partial x^\epsilon} = 0.$$

Since the condition $n \geqq 3$ is satisfied we therefore have $\mu = \text{const}$. This gives us the following theorem due to Schur(8).

THEOREM. *If the curvature of a metric space ($n \geqq 3$) is independent of the orientation at each point of the space, it does not vary from point to point.*

A metric space ($n \geqq 3$) of this sort is called a *space of constant curvature*; in particular we will say that a two-dimensional space has constant curvature if the Gaussian curvature of the space is constant, i.e. does not vary from point to point.*

Now let us determine the curvature of a space $\mathfrak{M}$ whose fundamental quadratic form is given by the expression

$$ds^2 = \frac{(dx^1)^2 + \ldots + (dx^n)^2}{\left[1 + \frac{\mu}{4}(x^{1^2} + \ldots + x^{n^2})\right]^2}, \tag{50.12}$$

where μ is a constant. Then

$$\Gamma_{\alpha\beta\gamma} = \frac{-\mu/2}{\left[1 + \frac{\mu}{4}\Sigma(x^\nu)^2\right]^3}(\delta^\alpha_\gamma x^\beta + \delta^\alpha_\beta x^\gamma - \delta^\beta_\gamma x^\alpha),$$

and

$$\Gamma^\alpha_{\beta\gamma} = \frac{-\mu/2}{\left[1 + \frac{\mu}{4}\Sigma(x^\nu)^2\right]}(\delta^\alpha_\gamma x^\beta + \delta^\alpha_\beta x^\gamma - \delta^\beta_\gamma x^\alpha).$$

From (12.14) and (12.20) we see that

$$B_{\alpha\beta\gamma\delta} = g_{\alpha\nu}\left(\frac{\partial \Gamma^{\nu}_{\beta\gamma}}{\partial x^{\delta}} - \frac{\partial \Gamma^{\nu}_{\beta\delta}}{\partial x^{\gamma}}\right) + \Gamma_{\alpha\sigma\delta}\,\Gamma^{\sigma}_{\beta\gamma} - \Gamma_{\alpha\sigma\gamma}\,\Gamma^{\sigma}_{\beta}$$

$$= \frac{1}{2}\left(\frac{\partial^2 g_{\alpha\gamma}}{\partial x^{\beta}\,\partial x^{\delta}} - \frac{\partial^2 g_{\beta\gamma}}{\partial x^{\alpha}\,\partial x^{\delta}} - \frac{\partial^2 g_{\alpha\delta}}{\partial x^{\beta}\,\partial x^{\gamma}} + \frac{\partial^2 g_{\beta\delta}}{\partial x^{\alpha}\,\partial x^{\gamma}}\right) + \Gamma_{\nu\alpha\gamma}\,\Gamma^{\nu}_{\beta\delta} - \Gamma_{\nu\alpha\delta}\,\Gamma^{\nu}_{\beta\gamma}.$$

Hence, substituting the special values of the $g_{\alpha\beta}$ determined by (50.12), we obtain

(50.13) $$B_{\alpha\beta\gamma\delta} = \frac{\quad}{\left[1 + \frac{\mu}{4}\Sigma(x^{\nu})^2\right]^4}(\delta^{\alpha}_{\gamma}\,\delta^{\beta}_{\delta} - \delta^{\alpha}_{\delta}\,\delta^{\beta}_{\gamma}).$$

But $$g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma} = \frac{\delta^{\alpha}_{\gamma}\,\delta^{\beta}_{\delta} - \delta^{\alpha}_{\delta}\,\delta^{\beta}_{\gamma}}{\left[1 + \frac{\mu}{4}\Sigma(x^{\nu})^2\right]^4}$$

Hence we obtain the equations (50.9) and it follows that the curvature at every point P of the space is independent of the orientation, i.e. *$\mathfrak{M}$ is a space of constant curvature.*

It will be shown in § 86 that a transformation exists relating the coordinates of the neighbourhood of an arbitrary point P of a space of constant curvature with the points of a neighbourhood of an arbitrary point P' of another space of the same constant curvature in such a way that the fundamental quadratic differential forms of the two spaces are transformable into one another. Hence, *the quadratic differential form of a Riemannian space of constant curvature μ can always be transformed into the form* (50.12) *in a sufficiently small neighbourhood of an arbitrary point P.*

More generally it can be shown that the fundamental quadratic form of a metric space $\mathcal{T}$ of constant curvature μ is transformable into

$$ds^2 = \frac{e_1(dx^1)^2 + \ldots + e_n(dx^n)^2}{\left[1 + \frac{\mu}{4}(e_1 x^{1^2} + \ldots + e_n x^{n^2})\right]^2},$$

where e_i is either $+1$ or -1.

51. Identities in the components of the projective curvature tensor

In § 18 we noted certain of the identities satisfied by the components of the projective curvature tensor. We shall now consider the problem of determining the complete sets of identities in the components of this tensor.

Let us define the quantities

(51.1) $$Q^{i}_{jkl\ldots m} = \left(\frac{\partial^m \Xi^{i}_{jk}}{\partial \eta^l \ldots \partial \eta^m}\right)_0$$

with reference to the notation of § 31; these quantities $Q^{i}_{jkl\ldots m}$ are expressible as functions of the coordinates x^i by formulae analogous to those for the components of the normal tensors and obtainable in a similar manner(9). On account of the equations (31.1) and (31.30), the symmetry of the components Ξ^{i}_{jk}, and their definition by means of (51.1), the quantities $Q^{i}_{jkl\ldots m}$ satisfy the following identities

(51.2) $$Q^{i}_{jkl\ldots m} = Q^{i}_{kjl\ldots m} = Q^{i}_{jkr\ldots s}, \quad Q^{i}_{ikl\ldots m} = 0, \quad S(Q^{i}_{jkl\ldots m}) = 0,$$

where $r, \ldots, s$ denotes any permutation of the indices $l, \ldots, m$ and S has a significance exactly analogous to that in (41.2). It is evident on the basis of

the procedure of § 41 that *the identities* (51.2) *constitute a complete set of identities for the quantities* $Q^i_{jkl\ldots m}$.

In particular (51.2) gives

$$(51.3) \qquad Q^i_{jkl} = Q^i_{kjl}, \quad Q^i_{jkl} + Q^i_{klj} + Q^i_{ljk} = 0, \quad Q^i_{ikl} = 0,$$

as the complete sets of identities of the quantities Q^i_{jkl}.

We know that the components Ξ^i_{jk} or the components $\star C^i_{jk}$ vanish at $\eta^i = 0$ in consequence of the equations (31.1); similarly the components $\star C^0_{jk}$ vanish at the origin of the projective normal coordinate system in consequence of the equations of definition of these components and the above identities (51.3). There remain only the components $\star C^\alpha_{\beta 0}$ or $\star C^\alpha_{0\beta}$ which have the values $-\delta^\alpha_\beta/(n+1)$ throughout the projective normal coordinate system. With these facts in mind, let us transform the components W^i_{jkl} of the projective-affine curvature tensor defined in § 18 to the projective normal coordinate system and evaluate at the origin of this system. We thereby obtain the equations

$$(51.4) \qquad W^i_{jkl} = Q^i_{jkl} - Q^i_{jlk}.$$

By the procedure of § 49 we can now combine the above equations (51.4) with the identities (51.3) to obtain

$$(51.5) \qquad 3Q^i_{jkl} = W^i_{jkl} + W^i_{kjl}.$$

It follows that the quantities Q^i_{jkl} are the components of a projective affine tensor.*

We can now easily verify that the following identities

$$(51.6) \qquad W^i_{jkl} = -W^i_{jlk}, \quad W^i_{jkl} + W^i_{klj} + W^i_{ljk} = 0, \quad W^i_{jki} = 0,$$

are satisfied. Also from (51.5) and (51.6) the equations (51.3) and (51.4) can be obtained. Hence, *the identities* (51.6) *are a complete set of identities of the components* W^i_{jkl} *of the projective-affine curvature tensor.*

Let us now transform the components $\star B^\alpha_{\beta\gamma\delta}$ to a projective normal coordinate system for the affine representation $A^\star_{n+1}$ and evaluate at the origin of this system. Then in addition to the identities (18.2) and (51.4) there will result

$$(51.7) \qquad \star B^0_{jkl} = \left(\frac{n+1}{n-1}\right)(Q^i_{jkli} - Q^i_{jlki}).$$

Combining these equations with the identities

$$(51.8) \qquad Q^i_{jkli} + Q^i_{klji} + Q^i_{ljki} = 0,$$

which follow readily from (51.2), we can obtain

$$(51.9) \qquad 3Q^i_{jkli} = \left(\frac{n-1}{n+1}\right)(\star B^0_{jkl} + \star B^0_{kjl});$$

* The quantities Q^i_{jklm}, Q^i_{jklmp}, ... do not, however, constitute the components of a tensor; this follows from the fact that the relation between two systems of projective normal coordinates is not linear but linear-fractional. See (31.28).

also the identities

$$\text{(51.10)} \qquad {}^\star B^0_{jkl} = -{}^\star B^0_{jlk}, \quad {}^\star B^0_{jkl} + {}^\star B^0_{klj} + {}^\star B^0_{ljk} = 0,$$

are a direct result of (51.7) and (51.8).

Now the identities (51.3) and (51.8) and the relations (51.4) and (51.7) follow from the identities (51.6) and (51.10) and relations (51.5) and (51.9). This proves that *the identities*

$$\text{(51.11)} \qquad \begin{cases} {}^\star B^\alpha_{0\gamma\delta} = {}^\star B^\alpha_{\beta 0\delta} = {}^\star B^\alpha_{\beta\gamma 0} = 0, \\ {}^\star B^\alpha_{jkl} + {}^\star B^\alpha_{klj} + B^\alpha_{ljk} = 0, \\ {}^\star B^\alpha_{jkl} = -{}^\star B^\alpha_{jlk}, \quad {}^\star B^i_{jki} = 0, \end{cases}$$

constitute a complete set of identities of the components ${}^\star B^\alpha_{\beta\gamma\delta}$ *of the projective curvature tensor.*

52. Certain divergence identities

Considering the identities (49.13) in the special case of the metric space, let us put $\alpha = \epsilon$ and sum on these indices; there results

$$\text{(52.1)} \qquad B^\alpha_{\beta\gamma\delta,\alpha} + B_{\beta\delta,\gamma} - B_{\beta\gamma,\delta} = 0,$$

where we put

$$B_{\beta\delta} = B^\alpha_{\beta\delta\alpha}.$$

Now multiply (52.1) by $g^{\beta\delta}$ and sum on the indices β and δ. The identities obtained can easily be given the form (10)

$$\text{(52.2)} \qquad B^\alpha_{\beta,\alpha} = \frac{1}{2}\frac{\partial B}{\partial x^\beta},$$

where

$$B^\alpha_\beta = \qquad\qquad {}_\delta, \quad B = g^{\alpha\beta} B_{\alpha\beta}.$$

In other words, *the divergence of the tensor B having the components*

$$B^\alpha_\beta - \tfrac{1}{2}\delta^\alpha_\beta B$$

*vanishes identically.**

Other identities corresponding to (52.2) can obviously be constructed. Without entering upon a general discussion of this problem, let us seek to find a set of identities analogous to (52.2) for the case of the space of distant parallelism. When dealing with quantities of the nature of absolute invariants it is by no means obvious how the operation of *divergence* can best be defined. However, actual investigation of the identities of the space of distant parallelism points to the following definition of the divergence as one permitting the construction of a set of invariants analogous to the above tensor B.

* The Einstein equations of the gravitational field are obtained by putting

$$B^\alpha_\beta - \tfrac{1}{2}\delta^\alpha_\beta = T^\alpha_\beta,$$

where the quantities T^α_β are the components of the energy tensor T which vanishes in regions free of matter. Since the divergence of the above tensor B vanishes, we have

$$T^\alpha_{\beta,\alpha} = 0;$$

these equations express the law of conservation of energy of the gravitational field.

Divergence rule: The formula

$$\nabla_m T^i_{j\dots m} = \sum_{m=1}^{n} e_m [T^i_{j\dots m,m} + 2h^i_{m,r} T^r_{j\dots m}]$$

defines the divergence of the set of scalars $T^i_{j\dots m}$ with respect to the index m. Any covariant index, i.e. covariant under transformations of the fundamental vectors, can of course appear in place of the particular index m in the definition of the divergence operation. A generalization of the above formula to include sets of scalars T with any number of contravariant indices can obviously be made.

Let us denote the quantities $\Lambda^\alpha_{\beta\gamma\delta}$ by λ^i_{jkl} with reference to a system of absolute normal coordinates and define the invariants $\mathfrak{A}^i_{jklm}$ by the formula

$$\mathfrak{A}^i_{jklm} = \left(\frac{\partial \lambda^i_{jkl}}{\partial z^m}\right)_{z=0}.$$

Now transform equations (35.14) to absolute normal coordinates, differentiate, and evaluate at the origin of the local system; we thus obtain a set of identities which can be written

$$\begin{aligned}(52.3)\qquad h^i_{j,kl,m} = h^i_{j,klm} - \mathfrak{A}^i_{jklm} + \tfrac{1}{2}h^i_{j,t}h^t_{m,kl} + \tfrac{1}{2}h^i_{t,k}h^t_{m,jl} \\ + \tfrac{1}{2}h^i_{t,l}h^t_{m,jk} + h^t_{m,j}h^i_{t,kl} + h^t_{m,k}h^i_{j,tl} + h^t_{m,l}h^i_{j,kt}.\end{aligned}$$

Interchanging the indices j, k in (52.3) and subtracting the resulting identities from (52.3), we obtain

$$\begin{aligned}(52.4)\qquad h^i_{j,kl,m} - h^i_{k,jl,m} = h^i_{j,klm} - h^i_{k,jlm} + h^i_{j,t}h^t_{m,kl} + h^i_{t,k}h^t_{m,jl} \\ + h^t_{m,j}(h^i_{t,kl} - h^i_{k,tl}) + h^t_{m,k}(h^i_{j,tl} - h^i_{t,jl}) + h^t_{m,l}(h^i_{j,kt} - h^i_{k,jt}).\end{aligned}$$

To the identities (52.4) we now add those two sets of identities which result from (52.4) by permutating the indices k, l, m cyclically. Interchanging k and l in the identities so obtained and again adding, we have a set of identities which can be reduced to the form

$$\begin{aligned}(52.5)\qquad 8h^i_{j,klm} = 3[h^i_{j,kl,m} + h^i_{j,lm,k} + h^i_{j,mk,l} + h^t_{j,m}h^i_{t,kl} + h^t_{j,l}h^i_{t,mk} + h^t_{j,k}h^i_{t,lm}] \\ + h^i_{t,k}h^t_{j,lm} + h^i_{t,l}h^t_{j,km} + h^i_{t,m}h^t_{j,kl}.\end{aligned}$$

These identities express the invariants $h^i_{j,klm}$ in terms of the invariants $h^i_{j,kl,m}$ plus invariants of lower order in the derivatives of the h^i_α.

To obtain the inverse form of the set of identities (52.5) we deduce the formula

$$\mathfrak{A}^i_{jklm} = -\tfrac{1}{3}h^i_{m,jkl} + \tfrac{1}{6}(h^i_{t,j}h^t_{m,kl} + h^i_{t,k}h^t_{m,lj} + h^i_{t,l}h^t_{m,jk}),$$

which we use to eliminate the invariants $\mathfrak{A}^i_{jklm}$ from (52.3). The result of this elimination is a set of identities which can be put into the form

$$\begin{aligned}(52.6)\qquad 3h^i_{j,kl,m} = 3h^i_{j,klm} + h^i_{m,jkl} + 2h^i_{j,t}h^t_{m,kl} + h^i_{t,k}h^t_{m,jl} \\ + h^i_{t,l}h^t_{m,jk} + 3(h^t_{m,j}h^i_{t,kl} + h^t_{m,k}h^i_{j,tl} + h^t_{m,l}h^i_{j,kt}).\end{aligned}$$

Finally the set of identities

$$\begin{aligned}(52.7)\qquad 5h^i_{j,kl,m} = 3(h^i_{j,lm,k} + h^i_{j,mk,l}) + h^i_{m,kl,j} + h^i_{m,lj,k} + h^i_{m,jk,l} \\ + h^i_{t,l}(3h^t_{m,jk} + h^t_{j,km}) + h^i_{t,k}(3h^t_{m,jl} + h^t_{j,lm}) \\ + 5h^i_{j,t}h^t_{m,kl} + h^i_{t,m}h^t_{j,kl} + 6h^t_{m,j}h^i_{t,kl} + h^t_{m,k}(8h^i_{j,tl} + h^i_{t,jl}) \\ + h^t_{m,l}(8h^i_{j,kt} + h^i_{t,jk}) + 3(h^t_{j,l}h^i_{t,mk} + h^t_{j,k}h^i_{t,lm})\end{aligned}$$

can be obtained by eliminating the invariants $h^i_{j,klm}$ between the identities (52.5) and (52.6).

By use of (48.2) and (42.4) it readily follows that

$$(52.8)\qquad \nabla_k h^i_{j,k} = \tfrac{3}{4}\sum e_k(h^i_{j,kk} + 4h^i_{r,k}h^r_{k,j})$$

identically. Now put $l=k$ and $m=j$ in (52.7), then multiply these equations through by $e_j e_k$ and sum on the two repeated indices. This gives

$$\sum_{j=1}^{n}\sum_{k=1}^{n} e_j e_k (h^i_{j,kk,j} + h^i_{r,j} h^r_{j,kk} + 2h^r_{k,j} h^i_{j,kr}) = 0 \tag{52.9}$$

identically. We next consider the set of identities

$$2\sum_{j=1}^{n}\sum_{k=1}^{n} e_j e_k h^r_{k,j} h^i_{j,kr} = 4\sum_{j=1}^{n}\sum_{k=1}^{n} e_j e_k (h^i_{r,k} h^r_{k,j})_{,j} + 3\sum_{j=1}^{n}\sum_{k=1}^{n} e_j e_k h^i_{j,m}(h^m_{j,kk} + \tfrac{8}{3} h^m_{r,k} h^r_{k,j}),$$

which we use to eliminate the last set of terms from (52.9). As a result we obtain the set of identities

$$\sum_{j=1}^{n} e_j \left\{ \left[\sum_{k=1}^{n} e_k (h^i_{j,kk} + 4h^i_{r,k} h^r_{k,j}) \right]_{,j} + 2h^i_{j,m} \left[\sum_{k=1}^{n} e_k (h^m_{j,kk} + 4h^i_{r,k} h^r_{k,j}) \right] \right\} = 0.$$

By (52.8) this last set of identities can be given the form

$$\nabla_j \nabla_k h^i_{j,k} = 0, \tag{52.10}$$

i.e. *the divergence of the set of scalars* $\nabla_k h^i_{j,k}$ *vanishes identically.*

Since the scalars $\nabla_k h^i_{j,k}$ define the divergence of the $h^i_{j,k}$, we can express the above result in obvious terms by saying that *the second divergence of the set of scalars* $h^i_{j,k}$ *vanishes identically.*

53. A GENERAL METHOD FOR OBTAINING DIVERGENCE IDENTITIES

Let us consider any scalar function of the $g_{\alpha\beta}$ and their derivatives up to any finite order r,

$$K(g_{\alpha\beta};\ g_{\alpha\beta|\gamma};\ \ldots;\ g_{\alpha\beta|\gamma\ldots\epsilon}),$$

which is analytic in its arguments throughout a closed n-dimensional domain $\mathscr{D}$. Here we have set

$$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = g_{\alpha\beta|\gamma};\quad \frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^\delta} = g_{\alpha\beta|\gamma\delta};\quad \ldots,$$

and it is assumed that these quantities are analytic functions of the x^ν throughout the domain $\mathscr{D}$. Then, denoting the determinant $|g_{\alpha\beta}|$ by g, the integral

$$I = \int K\sqrt{g}\,d\tau, \tag{53.1}$$

extended over the domain $\mathscr{D}$, is invariant, i.e. its value is unchanged by an analytic transformation of the coordinates of $\mathscr{D}$.

Now vary the $g_{\alpha\beta}$ so as to obtain

$$g'_{\alpha\beta} = g_{\alpha\beta} + \epsilon h_{\alpha\beta}, \tag{53.2}$$

where the $h_{\alpha\beta}$ are the components of an arbitrary tensor which are analytic in the x^ν and which vanish, together with their first $r-1$ derivatives, on the boundary of the domain $\mathscr{D}$; also ϵ is an infinitesimal. Then

$$\begin{aligned} I' &= \int K'\sqrt{g'}\,d\tau \\ &= \int \left\{ K\sqrt{g} + \epsilon \left[\frac{\partial(K\sqrt{g})}{\partial g_{\alpha\beta}} h_{\alpha\beta} + \ldots + \frac{\partial(K\sqrt{g})}{\partial g_{\alpha\beta|\gamma\ldots\epsilon}} h_{\alpha\beta|\gamma\ldots\epsilon} \right] + \ldots \right\} d\tau, \end{aligned}$$

where the dots denote terms of higher order in ϵ. Hence

$$\lim_{\epsilon\to 0}\left(\frac{I'-I}{\epsilon}\right) = \int \left[\frac{\partial(K\sqrt{g})}{\partial g_{\alpha\beta}} h_{\alpha\beta} + \ldots + \frac{\partial(K\sqrt{g})}{\partial g_{\alpha\beta|\gamma\ldots\epsilon}} h_{\alpha\beta|\gamma\ldots\epsilon} \right] d\tau. \tag{53.3}$$

Now
$$\int \frac{\partial (K\sqrt{g})}{\partial g_{\alpha\beta\,|\,\gamma\ldots\delta\epsilon}} \frac{\partial^s h_{\alpha\beta}}{\partial x^\gamma \ldots \partial x^\delta \partial x^\epsilon} d\tau + \int \frac{\partial}{\partial x^\epsilon} \left(\frac{\partial (K\sqrt{g})}{\partial g_{\alpha\beta\,|\,\gamma\ldots\delta\epsilon}} \right) \frac{\partial^{s-1} h_{\alpha\beta}}{\partial x^\gamma \ldots \partial x^\delta} d\tau$$
$$= \int \frac{\partial}{\partial x^\epsilon} \left[\frac{\partial (K\sqrt{g})}{\partial g_{\alpha\beta\,|\,\gamma\ldots\delta\epsilon}} \frac{\partial^{s-1} h_{\alpha\beta}}{\partial x^\gamma \ldots \partial x^\delta} \right] dx^1 \ldots dx^n$$
$$= \sum_{\epsilon=1}^{n} \int \left[\frac{\partial (K\sqrt{g})}{\partial g_{\alpha\beta\,|\,\gamma\ldots\delta\epsilon}} \frac{\partial^{s-1} h_{\alpha\beta}}{\partial x^\gamma \ldots \partial x^\delta} \right] dx^1 \ldots dx^{\epsilon-1} dx^{\epsilon+1} \ldots dx^n$$

over the boundary of the domain $\mathcal{D}$; hence the left member of this equation is zero for $s=1, \ldots, r$ by our hypothesis that the $h_{\alpha\beta}$ and their first $r-1$ derivatives vanish on the boundary. By repeating this integration by parts s times, we obtain finally

$$\int \frac{\partial (K\sqrt{g})}{\partial g_{\alpha\beta\,|\,\gamma\ldots\epsilon}} \frac{\partial^s h_{\alpha\beta}}{\partial x^\gamma \ldots \partial x^\epsilon} d\tau = (-1)^s \int \frac{\partial^s}{\partial x^\gamma \ldots \partial x^\epsilon} \left(\frac{\partial (K\sqrt{g})}{\partial g_{\alpha\beta\,|\,\gamma\ldots\epsilon}} \right) h_{\alpha\beta}\, d\tau. \tag{53.4}$$

Substituting (53.4) into (53.3), we have

$$\lim_{\epsilon\to 0} \left(\frac{I'-I}{\epsilon} \right) = \int P^{\alpha\beta} h_{\alpha\beta} \sqrt{g}\, d\tau, \tag{53.5}$$

where we have set

$$P^{\alpha\beta} = \frac{1}{\sqrt{g}} \left\{ \frac{\partial (K\sqrt{g})}{\partial g_{\alpha\beta}} - \frac{\partial}{\partial x^\gamma} \left(\frac{\partial (K\sqrt{g})}{\partial g_{\alpha\beta\,|\,\gamma}} \right) + \ldots + (-1)^r \frac{\partial^r}{\partial x^\gamma \ldots \partial x^\epsilon} \left(\frac{\partial (K\sqrt{g})}{\partial g_{\alpha\beta\,|\,\gamma\ldots\epsilon}} \right) \right\}. \tag{53.6}$$

It is obvious because of the symmetry of the $g_{\alpha\beta}$ that $P^{\alpha\beta}$ is symmetrical in the indices α and β.

Now since I and I' are both invariants, the integral in (53.5) must likewise be invariant under coordinate transformations. Hence using the formula for the transformation of the components $h_{\alpha\beta}$, and making use of the fact that $\sqrt{g}\,d\tau$ is invariant under coordinate transformations, we see that

$$\int \left[\bar{P}^{\mu\nu} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} - P^{\alpha\beta} \right] h_{\alpha\beta} \sqrt{g}\, d\tau = 0.$$

Now suppose that there were a point P in the domain $\mathcal{D}$ at which one of the expressions in the brackets did not vanish for a particular pair of values of α and β. Then, since this expression is continuous, there will be a certain domain Δ of the point P in which it does not vanish. We may then choose our domain $\mathcal{D}$ to coincide with Δ, and since the $h_{\alpha\beta}$ are arbitrary within $\mathcal{D}$ we will choose $h_{\alpha\beta}$ to have the same sign as the bracket expression for this pair of values of α and β, and to be zero for all other values of α and β. Then the integrand is positive within $\mathcal{D}$ so that the integral cannot vanish. Hence our assumption that the bracket expressions could be different from zero at any point is false, i.e. the $P^{\alpha\beta}$ have the tensor law of transformation.

Now let us consider an infinitesimal transformation of coordinates

$$\bar{x}^\nu = x^\nu + \epsilon y^\nu(x), \tag{53.7}$$

where the y^ν are analytic functions of the x^ν subject only to the condition that they and their derivatives up to the rth order shall vanish on the boundary of the domain $\mathcal{D}$.*
From (9.7) we see that the components of the fundamental tensor must transform by the equations

$$g_{\alpha\beta}(x) = \bar{g}_{\mu\nu}(\bar{x}) \left[\delta^\mu_\alpha \delta^\nu_\beta + \epsilon \left(\delta^\mu_\alpha \frac{\partial y^\nu}{\partial x^\beta} + \delta^\nu_\beta \frac{\partial y^\mu}{\partial x^\alpha} \right) + \epsilon^2 \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial y^\nu}{\partial x^\beta} \right],$$

or
$$g_{\alpha\beta}(x) = \bar{g}_{\alpha\beta}(\bar{x}) + \epsilon \left(g_{\alpha\nu} \frac{\partial y^\nu}{\partial x^\beta} + g_{\nu\beta} \frac{\partial y^\nu}{\partial x^\alpha} \right) + \text{terms of higher order}$$
$$= \bar{g}_{\alpha\beta}(\bar{x}) - \epsilon \phi_{\alpha\beta}(x) + \ldots. \tag{53.8}$$

* It is here assumed that $\mathcal{D}$ is sufficiently restricted so that (53.7) has a unique inverse in $\mathcal{D}$ and hence represents a transformation of the coordinates of the domain $\mathcal{D}$ (see § 1).

If we carry out on the integral I the transformation of coordinates (53.7), we have

$$\bar{I}=\int\bar{K}\sqrt{\bar{g}}\,d\bar{\tau}$$
$$=\int\left\{K+\epsilon\left[\left(\frac{\partial\bar{K}}{\partial\bar{g}_{\alpha\beta}}\right)_{\epsilon=0}\left(\frac{d\bar{g}_{\alpha\beta}}{d\epsilon}\right)_{\epsilon=0}+\cdots+\left(\frac{\partial\bar{K}}{\partial\bar{g}_{\alpha\beta\,|\,\gamma\cdots\epsilon}}\right)_{\epsilon=0}\left(\frac{d\bar{g}_{\alpha\beta\,|\,\gamma\cdots\epsilon}}{d\epsilon}\right)_{\epsilon=0}\right]+\cdots\right\}\sqrt{g}\,d\tau.$$

By differentiating (53.8) s times with respect to $\bar{x}^\gamma, \ldots, \bar{x}^\delta, \bar{x}^\eta, \bar{x}^\epsilon$, then differentiating with respect to ϵ, and evaluating at $\epsilon=0$, it can easily be seen that

$$\left(\frac{d\bar{g}_{\alpha\beta\,|\,\gamma\cdots\delta\eta\epsilon}}{d\epsilon}\right)_{\epsilon=0}=\phi_{\alpha\beta\,|\,\gamma\cdots\epsilon}-P\left(g_{\alpha\beta\,|\,\lambda\gamma\cdots\delta\eta}\frac{\partial y^\lambda}{\partial x^\epsilon}\right)$$
$$-P\left(g_{\alpha\beta\,|\,\lambda\gamma\cdots\delta}\frac{\partial^2 y^\lambda}{\partial x^\eta\partial x^\epsilon}\right)-\cdots-g_{\alpha\beta\,|\,\lambda}\frac{\partial^s y^\lambda}{\partial x^\gamma\ldots\partial x^\epsilon},$$

where $P(\quad)$ represents the sum of all terms obtainable from the one in parenthesis by taking for the indices on the derivative appearing in the term all possible combinations of the indices $\gamma, \ldots, \epsilon$. Also

$$\left(\frac{\partial\bar{K}}{\partial\bar{g}_{\alpha\beta}}\right)_{\epsilon=0}=\left(\frac{\partial K}{\partial g_{\alpha\beta}}\right);\quad\left(\frac{\partial\bar{K}}{\partial\bar{g}_{\alpha\beta\,|\,\gamma\cdots\epsilon}}\right)_{\epsilon=0}=\frac{\partial K}{\partial g_{\alpha\beta\,|\,\gamma\cdots\epsilon}}.$$

Since $\bar{I}=I$, we have

$$\lim_{\epsilon\to 0}\left(\frac{\bar{I}-I}{\epsilon}\right)=0.$$

Hence

$$(53.9)\qquad\int\left\{\frac{\partial K}{\partial g_{\alpha\beta}}\phi_{\alpha\beta}+\frac{\partial K}{\partial g_{\alpha\beta\,|\,\gamma}}\left(\phi_{\alpha\beta\,|\,\gamma}-\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}\frac{\partial y^\lambda}{\partial x^\gamma}\right)+\cdots+\frac{\partial K}{\partial g_{\alpha\beta\,|\,\gamma\cdots\epsilon}}\left(\quad\right)\right\}\sqrt{g}\,d\tau=0.$$

Now the integral

$$(53.10)\qquad\int\frac{\partial}{\partial x^\lambda}(K\sqrt{g}\,y^\lambda)\,d\tau$$

vanishes because of our assumption that $y^\nu=0$ on the boundary of $\mathcal{V}$. In expanding this integral we shall make use of the identity

$$(53.11)\qquad\sqrt{g}\frac{\partial y^\lambda}{\partial x^\lambda}=\frac{\partial\sqrt{g}}{\partial g_{\alpha\beta}}\left(g_{\alpha\lambda}\frac{\partial y^\lambda}{\partial x^\beta}+g_{\lambda\beta}\frac{\partial y^\lambda}{\partial x^\alpha}\right),$$

which can be proved as follows. We have

$$\frac{\partial\log g}{\partial g_{\alpha\beta}}=g^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial g_{\alpha\beta}}=g^{\alpha\beta}.$$

Hence
$$g_{\alpha\lambda}\frac{\partial\sqrt{g}}{\partial g_{\alpha\beta}}=\tfrac{1}{2}\sqrt{g}\,g_{\alpha\lambda}\frac{\partial\log g}{\partial g_{\alpha\beta}}=\tfrac{1}{2}\sqrt{g}\,\delta^\beta_\lambda,$$

and
$$g_{\lambda\beta}\frac{\partial\sqrt{g}}{\partial g_{\alpha\beta}}=\tfrac{1}{2}\sqrt{g}\,g_{\beta\lambda}\frac{\partial\log g}{\partial g_{\alpha\beta}}=\tfrac{1}{2}\sqrt{g}\,\delta^\alpha_\lambda.$$

Multiplying these two latter equations by $\dfrac{\partial y^\lambda}{\partial x^\beta}$ and $\dfrac{\partial y^\lambda}{\partial x^\alpha}$, respectively, and adding the resulting equations give us (53.11). By expanding (53.10) and making use of (53.11) we get the equation

$$(53.12)\qquad 0=\int\left[\frac{\partial K}{\partial g_{\alpha\beta}}\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}y^\lambda+\cdots+\frac{\partial K}{\partial g_{\alpha\beta\,|\,\gamma\cdots\epsilon}}\frac{\partial g_{\alpha\beta\,|\,\gamma\cdots\epsilon}}{\partial x^\lambda}y^\lambda\right]\sqrt{g}\,d\tau$$
$$+\int K\frac{\partial\sqrt{g}}{\partial g_{\alpha\beta}}\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}y^\lambda d\tau+\int K\frac{\partial\sqrt{g}}{\partial g_{\alpha\beta}}\left(g_{\alpha\lambda}\frac{\partial y^\lambda}{\partial x^\beta}+g_{\lambda\beta}\frac{\partial y^\lambda}{\partial x^\alpha}\right)d\tau.$$

By subtracting (53.12) from (53.9) and substituting the value of $\phi_{\alpha\beta}$ from (53.8), we obtain the equation

$$\int\left\{-\left(\sqrt{g}\frac{\partial K}{\partial g_{\alpha\beta}}+K\frac{\partial\sqrt{g}}{\partial g_{\alpha\beta}}\right)\left(\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}y^\lambda+g_{\alpha\lambda}\frac{\partial y^\lambda}{\partial x^\beta}+g_{\lambda\beta}\frac{\partial y^\lambda}{\partial x^\alpha}\right)\right.$$
$$\left.-\sqrt{g}\frac{\partial K}{\partial g_{\alpha\beta\,|\,\gamma}}\left(-\phi_{\alpha\beta\,|\,\gamma}+\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}\frac{\partial y^\lambda}{\partial x^\gamma}+\frac{\partial g_{\alpha\beta\,|\,\gamma}}{\partial x^\lambda}y^\lambda\right)-\cdots\right\}d\tau=0$$

or, since $\sqrt{g}$ is not a function of the derivatives of $g_{\alpha\beta}$, we have

$$\int\left\{\frac{\partial(K\sqrt{g})}{\partial g_{\alpha\beta}}h_{\alpha\beta}+\frac{\partial(K\sqrt{g})}{\partial g_{\alpha\beta;\gamma}}h_{\alpha\beta|\gamma}+\ldots+\frac{\partial(K\sqrt{g})}{\partial g_{\alpha\beta|\gamma\ldots\epsilon}}h_{\alpha\beta|\gamma\ldots\epsilon}\right\}d\tau=0,$$

where $h_{\alpha\beta}$ now represents the expression

$$\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}y^\lambda+g_{\alpha\lambda}\frac{\partial y^\lambda}{\partial x^\beta}+g_{\beta\lambda}\frac{\partial y^\lambda}{\partial x^\alpha}.$$

Hence by carrying out as before an integration by parts, we have

$$\int P^{\alpha\beta}h_{\alpha\beta}\sqrt{g}\,d\tau=0,$$

where $P^{\alpha\beta}$ is the tensor introduced previously.

If we set
$$g_{\beta\gamma}P^{\alpha\beta}=P^\alpha_\gamma,$$
we have
$$\sqrt{g}\,P^{\alpha\beta}h_{\alpha\beta}=\sqrt{g}\,P^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}y^\lambda+2P^\beta_\lambda\frac{\partial y^\lambda}{\partial x^\beta}\sqrt{g}$$

$$=\sqrt{g}\,P^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}y^\lambda+2\frac{\partial}{\partial x^\beta}(\sqrt{g}\,y^\lambda P^\beta_\lambda)-2y^\lambda\frac{\partial}{\partial x^\beta}(\sqrt{g}\,P^\beta_\lambda).$$

Hence by carrying out an integration by parts we have

$$\int\left\{\tfrac{1}{2}P^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}-\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\beta}(\sqrt{g}\,P^\beta_\lambda)\right\}\sqrt{g}\,y^\lambda d\tau=0. \tag{53.13}$$

By making use of the symmetry of the $P^{\alpha\beta}$, the identities (13.15) and

$$\frac{\partial g_{\alpha\beta}}{\partial x^\lambda}=g_{\alpha\sigma}\Gamma^\sigma_{\lambda\beta}+g_{\beta\sigma}\Gamma^\sigma_{\lambda\alpha},$$

the integrand in (53.13) becomes

$$\sqrt{g}\,y^\lambda\left\{P^\alpha_\sigma\Gamma^\sigma_{\lambda\alpha}-P^\beta_\lambda\Gamma^\alpha_{\alpha\beta}-\frac{\partial P^\beta_\lambda}{\partial x^\beta}\right\}=-\sqrt{g}\,y^\lambda P^\beta_{\lambda,\beta}.$$

Hence
$$\int P^\beta_{\lambda,\beta}y^\lambda\sqrt{g}\,d\tau=0,$$
and by an argument similar to that used previously we conclude that

$$P^\beta_{\lambda,\beta}=0$$

throughout the domain $\mathscr{D}$. Since the first covariant derivative of $g_{\alpha\beta}$ vanishes identically we may also write

$$P^{\alpha\beta}{}_{,\beta}=0,$$

i.e. *the tensor P whose components are defined by* (53.6) *in the domain $\mathscr{D}$, is such that its divergence vanishes identically*(11).

If we take
$$K=g^{\beta\gamma}B^\sigma_{\beta\gamma\sigma}=B,$$
where the $B^\alpha_{\beta\gamma\delta}$ are the components of the curvature tensor, it follows from (53.6) that

$$P^\alpha_\beta=B^\alpha_\beta-\tfrac{1}{2}\delta^\alpha_\beta B;$$

hence as a special case of the result of this section we have that the divergence of the tensor B vanishes identically as proved in § 52.

54. Numbers of algebraically independent components of certain spatial invariants

Since the identities (41.1) and (41.2) constitute a complete set, it follows that the enumeration of the independent components of any normal tensor A from (41.1) and (41.2) will yield the total number of algebraically independent components of the normal tensor A. Let $A(n,p)$ stand for the number of algebraically independent components $A^\alpha_{\beta\gamma\delta\ldots\sigma}$ with $p+2$ sub-

scripts $\beta, \gamma, \delta, \ldots, \sigma$. Then if $K(\mu,\lambda)$ denotes the number of combinations with repetitions of μ things taken λ at a time, we shall have

$$A(n,p) = n[K(n,2)K(n,p) - K(n,p+2)], \tag{54.1}$$

where

$$K(\mu,\lambda) = \frac{\mu(\mu+1)\ldots(\mu+\lambda-1)}{\lambda!}, \quad K(\mu,0)=1. \tag{54.2}$$

Similarly we have in consequence of (41.10) and (41.11) that the number of algebraically independent components $g_{\alpha\beta,\gamma\ldots\delta}$, where the number of subscripts $\alpha, \beta, \gamma, \ldots, \delta$ is $p+2$, is given by

$$G(n,p) = K(n,2)K(n,p) - nK(n,p+1) \quad (p \geqq 1); \tag{54.3}$$

it should be observed that $G(n,1)=0$ corresponding to the fact that all components $g_{\alpha\beta,\gamma}$ vanish identically; also $G(n,0)=K(n,2)$ gives the number of independent components $g_{\alpha\beta}$ on account of (40.1). For the number of algebraically independent components $h^i_{j,k\ldots m}$, where the number of subscripts $j, k, \ldots, m$ is $p+1$, we have

$$H(n,p) = n^2K(n,p) - nK(n,p+1) \quad (p \geqq 1), \tag{54.4}$$

in consequence of (42.1) and (42.2).

Using the substitution (54.2), the formulae (54.1), (54.3) and (54.4) can be given the forms

$$A(n,p) = \frac{1}{2}\cdot\frac{(n+p-1)!\,np}{(n-2)!\,(p+2)!}[(n+2)(p+1)+2n] \quad (p \geqq 1),$$

$$G(n,p) = \frac{n}{2}\cdot\frac{(n+p-1)!}{(n-2)!}\cdot\frac{(p-1)}{(p+1)!} \quad (p \geqq 1),$$

$$H(n,p) = \frac{(n+p-1)!\,np}{(n-2)!\,(p+1)!} \quad (p \geqq 1).$$

Since the components $A^\alpha_{\beta\gamma\delta,\epsilon}$ are expressible in terms of the components $A^\alpha_{\beta\gamma\delta\epsilon}$ by (43.1) and the components $A^\alpha_{\beta\gamma\delta\epsilon}$ are expressible in terms of the components $A^\alpha_{\beta\gamma\delta,\epsilon}$ by (43.6), it follows that the number of independent components $A^\alpha_{\beta\gamma\delta,\epsilon}$ is equal to the number of independent components $A^\alpha_{\beta\gamma\delta\epsilon}$ of the normal tensor A. More generally, consider the two sets of components

$$A^\alpha_{\beta\gamma\delta};\ A^\alpha_{\beta\gamma\delta\epsilon};\ \ldots;\ A^\alpha_{\beta\gamma\delta\epsilon\ldots\sigma}, \tag{I}$$

$$A^\alpha_{\beta\gamma\delta};\ A^\alpha_{\beta\gamma\delta,\epsilon};\ \ldots;\ A^\alpha_{\beta\gamma\delta,\epsilon\ldots\sigma}, \tag{II}$$

which are connected by the formulae (43.7) and (43.9). Let us suppose that all dependent quantities have been eliminated from the equations (43.7) and (43.9) so that these equations express relations between the independent quantities of the two sets. Suppose that the number of independent components of the set (I) were greater than the number of independent components of the set (II). Then a certain number of the equations (43.9), less than the total number of these equations, can be solved for the independent components of the set (II). When these latter relations are used to eliminate

the quantities of the set (II) from the remaining equations (43.9) we obtain relations between certain of the independent quantities of the set (I), which is contrary to the hypothesis that these quantities are independent. In a similar manner the number of independent quantities of the set (II) can not be greater than that of the set (I). Hence the number of independent components of the sets (I) and (II) are equal. It follows immediately by a recurrence consideration that the number of independent components of corresponding tensors in the sets (I) and (II) are equal, i.e. the number of independent components $A^{\alpha}_{\beta\gamma\delta,\epsilon\ldots\sigma}$ with p subscripts $\delta, \epsilon, \ldots, \sigma$ is equal to the number of independent components $A^{\alpha}_{\beta\gamma\delta\epsilon\ldots\sigma}$ and is therefore given by $A(n,p)$.

In a similar manner it follows that the numbers of independent components $g_{\alpha\beta,\gamma\delta,\epsilon\ldots\sigma}$ with p subscripts in the set $\gamma, \delta, \epsilon, \ldots, \sigma$, and $h^i_{j,k,l\ldots m}$ with p subscripts in the set $k, l, \ldots, m$, are given by $G(n,p)$ and $H(n,p)$, respectively. It is evident that the numbers of independent components of many other invariants can be found in an analogous manner.

Use of the above formulae for $A(n,p)$ and $G(n,p)$ enables us to prove immediately that for $n=2$, the equations (46.5) must be satisfied identically in consequence of the functional relations (41.7); similarly we can prove that for $n=2$, (46.10) is satisfied identically in consequence of the functional relations (41.14) and (41.15). First consider the functions $A^{\alpha}_{\beta\gamma\delta}(y)$ with reference to normal coordinates y^{α}, and suppose that they satisfy the complete set of identities (41.7). The number $A(2,1)$, which is equal to 4, gives the number of independent components $A^{\alpha}_{\beta\gamma\delta}$. Now if (41.7) gave all the conditions on the functions $A^{\alpha}_{\beta\gamma\delta}(y)$, the number of independent components $A^{\alpha}_{\beta\gamma\delta,\epsilon\ldots\sigma}$ for r subscripts $\epsilon, \ldots, \sigma$ would be $4K(2,r)$ or $4(r+1)$; but actually the number of independent quantities $A^{\alpha}_{\beta\gamma\delta,\epsilon\ldots\sigma}$ is given by $A(2,r+1)$, which is equal to $4(r+1)$. Hence it follows that the equations (46.5) cannot produce additional conditions on the quantities $A^{\alpha}_{\beta\gamma\delta,\epsilon\ldots\sigma}$ over those which result from (41.7) by extension. An analogous consideration applies to the components $g_{\alpha\beta,\gamma\delta}(y)$ in two variables. Here the number of independent components $g_{\alpha\beta,\gamma\delta}$ is $G(2,2)$ or 1. Then, since $K(2,r)$ and $G(2,r+2)$ each have the value $r+1$, it follows that (41.14) and (41.15) give all the functional conditions on the components $g_{\alpha\beta,\gamma\delta}(y)$; the conditions (46.10) for $n=2$ must therefore be satisfied identically in consequence of (41.14) and (41.15) and the conditions obtainable from them by extension. We thus arrive at the following

THEOREM. *For $n=2$, the functions $A^{\alpha}_{\beta\gamma\delta}(y)$, each of which is analytic in the neighbourhood of the values $y^{\alpha}=0$, will be the components of an affine normal tensor in a system of normal coordinates if, and only if, the functions $A^{\alpha}_{\beta\gamma\delta}(y)$ satisfy* (41.7). *Similarly, for $n=2$, the functions $g_{\alpha\beta,\gamma\delta}(y)$, each of which is analytic in the neighbourhood of the values $y^{\alpha}=0$, will be the components of a metric normal tensor in normal coordinates if, and only if, the functions $g_{\alpha\beta,\gamma\delta}(y)$ satisfy* (41.14) *and* (41.15).

With the exception of the case $n=2$, the equations (46.5) and (46.10) furnish additional conditions over those which result by extension from the complete sets of identities of the components $A^{\alpha}_{\beta\gamma\delta}$ and $g_{\alpha\beta,\gamma\delta}$, respectively. The determination of the exact arbitrariness of the functions $A^{\alpha}_{\beta\gamma\delta}(y)$ or $g_{\alpha\beta,\gamma\delta}(y)$ in finite form for $n \geqq 3$ will be made in Chapter X on the basis of the numbers $A(n,p)$ and $G(n,p)$. For this purpose we must, however, derive a

new form for the expressions $A(n,p)$ and $G(n,p)$ which we shall now proceed to consider.

Observing that

$$A(n,p+r)=nK(n,2)\,K(n,p+r)-nK(n,p+r+2), \tag{54.5}$$

$$G(n,p+r)=K(n+2)\,K(n,p+r)-nK(n,p+r+1), \tag{54.6}$$

let us now determine numbers A_i and B_i, depending only on n and p, such that the above expressions for $A(n,p+r)$ and $G(n,p+r)$ can be written

$$A(n,p+r)=\sum_{i=0}^{n-1} A_i K(n-i,r), \tag{54.7}$$

$$G(n,p+r)=\sum_{i=0}^{n-1} B_i K(n-i,r), \tag{54.8}$$

for all integer values r. To derive (54.7) and (54.8) we make use of the formula

$$K(n,p)=K(n,p-1)+K(n-1,p),$$

by repeated application of which we have

$$K(n,r+1)=K(n,r)+K(n-1,r)+\ldots+K(2,r)+1,$$

or

$$K(n,r+1)=\sum_{i=0}^{n-1} K(n-i,r)=\sum_{\alpha=1}^{n} K(\alpha,r). \tag{54.9}$$

Hence
$$\begin{aligned} K(n,p+r)&=\sum_{\alpha=0}^{n-1} K(n-\alpha,p+r-1)\\ &=\sum_{\alpha=0}^{n-1}\sum_{\beta=0}^{n-\alpha-1} K(n-\alpha-\beta,p+r-2)\\ &=\sum_{\alpha=0}^{n-1}\ldots\sum_{\epsilon=0}^{n-\alpha-\ldots-\eta-1} K(n-\alpha-\ldots-\eta-\epsilon,r),\end{aligned}$$

where the indices $\alpha,\beta,\ldots,\eta,\epsilon$ are p in number. Each of the indices $\alpha,\beta,\ldots,\eta,\epsilon$ can take on values $0,1,\ldots,n-1$, subject to the condition that the sum $\alpha+\beta+\ldots+\eta+\epsilon$ has a value between zero and $n-1$ inclusive. The number of times that the sum $\alpha+\beta+\ldots+\eta+\epsilon$ takes on the value i $(=0,1,\ldots,n-1)$ is equal to the number of permutations with repetitions of the numbers $0,1,\ldots,i$ taken p at a time such that the sum of the numbers in any permutation is equal to i; let us denote the number of times that the sum

$$\alpha+\beta+\ldots+\eta+\epsilon$$

takes on the value $i\,(=0,1,\ldots,n-1)$ by $S(i+1,p-1)$. More generally $S(q+1,r-1)$ will denote the number of permutations with repetitions of $0,1,\ldots,i$ taken r at a time such that the sum of the numbers in any permutation is equal to $q\,(\leqq i)$. Now observe that $S(q+1,r-1)$ is equal also to the number of permutations with repetitions of $0,1,\ldots,i$ taken $r-1$ at a time such that the sum of the numbers in any permutation is $\leqq q$ for $q\leqq i$; this is seen immediately on account of the fact that from any permutation of this latter type one of the type which go to make up the sum $S(q+1,r-1)$ can

be obtained by the addition of one of the numbers 0, 1, ..., i, and the fact that in this way all permutations which make up the sum $S(q+1, r-1)$ can be constructed. Hence

$$S(q+1, r-1) = \sum_{\alpha=1}^{q-1} S(\alpha, r-2)$$

for $q \leqq i$. Using this formula as a recurrence formula, we have

$$S(i+1, p-1) = \sum_{\alpha=1}^{i+1} \sum_{\beta=1}^{\alpha} \dots \sum_{\mu=1}^{\nu} S(\mu, 1), \tag{54.10}$$

where the number of indices α, β, ..., μ is $p-2$. But from (54.9) an exactly similar expression can be obtained for $K(i+1, p-1)$ in which the $S(\mu, 1)$ in (54.10) is replaced by $K(\mu, 1)$. Since $S(\mu, 1)$ and $K(\mu, 1)$ are each equal to μ, it follows that $S(i+1, p-1)$ is equal to $K(i+1, p-1)$. The above expression for $K(n, p+r)$ can therefore be written

$$K(n, p+r) = \sum_{i=0}^{n-1} K(i+1, p-1)\, K(n-i, r). \tag{54.11}$$

Substituting this expression for $K(n, p+r)$ and a similar expression for $K(n, p+r+2)$ into (54.5) we obtain (54.7), where

$$A_i = nK(n, 2)\, K(i+1, p-1) - nK(i+1, p+1). \tag{54.12}$$

Similarly by making a substitution of the type (54.11) into (54.6) we deduce (54.8), where

$$B_i = K(n, 2)\, K(i+1, p-1) - nK(i+1, p). \tag{54.13}$$

Let us also observe that

$$A(n, p) = \sum_{i=0}^{n-1} A_i, \tag{54.14}$$

$$G(n, p) = \sum_{i=0}^{n-1} B_i. \tag{54.15}$$

These equations follow from (54.12) and (54.13) when use is made of (54.9). They may be considered as a special case of (54.7) and (54.8) for $r=0$.

In our later work in Chapter X we shall only need (54.7) for $p=1$ and (54.8) for $p=2$, i.e.

$$A(n, 1+r) = \sum_{i=0} A_i K(n-i, r), \tag{54.16}$$

$$G(n, 2+r) = \sum_{i=0}^{n-2} B_i K(n-i, r), \tag{54.17}$$

for which cases we have the special formulae

$$A_i = nK(n, 2) - nK(i+1, 2) \quad (p=1), \tag{54.18}$$

$$B_i = K(n, 2)\, K(i+1, 1) - nK(i+1, 2) \quad (p=2). \tag{54.19}$$

It is to be noted that $A_{n-1}=0$ and $B_{n-1}=0$ is a consequence of these latter formulae(12).

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(2) See T. Y. Thomas, ref. (2), Chapter V, p. 770.

(3) The results of §§43–45 were first published by T. Y. Thomas, "Determination of affine and metric spaces by their differential invariants", *Math. Ann.* **101** (1929), pp. 713–28.

(4) The theorems of §§46 and 47 were given by T. Y. Thomas, "The existence theorems in the problem...", *Amer. Journ. Math.* **52** (1930), pp. 225–50.

(5) See T. Y. Thomas, "On the unified field theory, II", *Proc. N.A.S.* **16** (1930), pp. 830–5.

(6) See ref. (4), Chapter I. The completeness proofs at the end of the section were given by T. Y. Thomas, ref. (1).

(7) This definition of curvature was first given by B. Riemann, *Gesammelte Mathematische Werke*, 2nd ed. (B. G. Teubner, 1892), pp. 279 and 403. The special form of the line element derived at the end of the section was also given by B. Riemann, *ibid.* p. 282.

(8) F. Schur, "Ueber den Zusammenhang der Räume...", *Math. Ann.* **27** (1886), p. 563.

(9) O. Veblen and J. M. Thomas, ref. (3), Chapter V. Also see T. Y. Thomas, "The replacement theorem...", *Ann. of Math.* (2), **28** (1926), pp. 549–61.

(10) This identity was given for $n=3$ by G. Ricci, "Sulle superficie geodetiche...", *Atti dei Lincei, Rend.* (5), **12**[1] (1903), pp. 409–20. It was contained implicitly in the paper by A. Einstein, "Zur allgemeinen Relativitätstheorie", *Sitzungsberichte Preuss. Akad.* (1915), pp. 778–86. T. Levi-Civita appears to have been the first to prove this identity for the general case by the use of Bianchi's identity; see "Sulla espressione analitica spettante al tensore gravitazionale nella teoria di Einstein", *Atti dei Lincei, Rend.* (5), **26**[1] (1917), pp. 381–91. The identities for the space of distant parallelism were first given by T. Y. Thomas, ref. (5).

(11) Cf. D. Hilbert, "Die Grundlagen der Physik", *Göttingen Nachrichten* (1915), pp. 395–407, and *Math. Ann.* **92** (1924), pp. 1–32. See also A. S. Eddington, *The Mathematical Theory of Relativity*, 2nd ed. (Cambridge Univ. Press, 1924), pp. 140–1, and R. Weitzenböck, *Invariantentheorie* (P. Noordhoff, 1923), pp. 374–7.

(12) T. Y. Thomas and A. D. Michal, ref. (9), Chapter V, and T. Y. Thomas, ref. (4).

CHAPTER VII

ABSOLUTE SCALAR DIFFERENTIAL INVARIANTS AND PARAMETERS

IN a space of distant parallelism absolute scalar differential invariants can readily be constructed by the process of extension developed in Chapter V. An analogous method for the construction of scalar differential invariants is not, however, possible in the general affinely connected space nor in the metric, conformal, projective and Weyl spaces. Now the scalar differential invariants of these generalized spaces can be considered to be defined by means of complete systems of linear partial differential equations. In the present chapter such systems of equations are found and used to determine the numbers of absolute scalar differential invariants and parameters of any order p. We have, however, limited our considerations to the case of the invariants and parameters of the metric space and the affinely connected space of symmetric affine connection;* it is evident that we can develop a closely analogous treatment of the differential invariants and parameters of other generalized spaces.

The determination of the systems of differential equations which define the scalar differential invariants and parameters has its basis in the theory of continuous groups. In the beginning of this chapter we have given in small print that portion of group theory which has application to the problem of the scalar differential invariants and parameters(1); we have then extended these considerations on groups in order to include the substance, at least, of the recent theory of the group space of an r-parameter continuous group(2).

55. ABSTRACT GROUPS

Consider a class of abstract objects, finite or infinite in number, and a law of combination of any two objects of the class which we will call the *product* of the objects. The abstract objects of the class will be referred to as its *elements*. We denote any element of the class by the symbol T_a and the product of two elements T_a and T_b by $T_a T_b$. The class of objects will be said to form an *abstract group* with respect to the law of combination in question if the following four conditions are satisfied.†

A. *If T_a and T_b are elements of the class, then $T_a T_b$ is uniquely defined and is an element of the class.*

B. *The associative law holds, thus*

$$T_a(T_b T_c) = (T_a T_b) T_c,$$

i.e. *the element which is the product of the elements T_a and $T_b T_c$ is the same as the element obtained as the product of the elements $T_a T_b$ and T_c.*

* It is therefore to be understood that we are dealing with an affinely connected space of this type without especial mention of this fact throughout the remainder of this chapter.

† Conditions C and D can, in fact, be replaced by $IT_a = T_a$ and $T_a^{-1} T_a = I$ respectively, since the relations $T_a I = T_a$ and $T_a T_a^{-1} = I$ can then be deduced.

C. *There is in the class an element I called the identity such that*

$$IT_a = T_a I = T_a$$

for every element T_a.

D. *For any element T_a of the class there is an element T_a^{-1} such that*

$$T_a^{-1} T_a = T_a T_a^{-1} = I.$$

A very simple example of a group is furnished by the two numbers -1, $+1$ where the law of combination is that of algebraic multiplication. It is readily verified that Condition A is satisfied; likewise Condition B is satisfied as this condition becomes identical with the ordinary associative law of multiplication. Condition C is evidently satisfied by taking $+1$ as the identity I. Finally Condition D holds when the following identification

$$T_a = T_a^{-1} = -1, \quad T_b = T_b^{-1} = +1$$

is made. An abstract group containing a finite number of elements, such as the one in this example, is called a *finite group*.

As an example of a group containing an infinite number of elements we have the set of all continuous coordinate transformations

$$T_a : x^\alpha = f^\alpha(\bar{x}^1, \ldots, \bar{x}^n) \tag{55.1}$$

of a definite space region $\mathscr{R}$. In fact consider the successive application of a particular transformation T_a and another transformation of the set

$$T_b : \bar{x}^\alpha = \phi^\alpha(\bar{\bar{x}}^1, \ldots, \bar{\bar{x}}^n),$$

and let us define the product $T_b T_a$ as the resultant of these two transformations, i.e.

$$T_b T_a : x^\alpha = f^\alpha(\phi^1(\bar{\bar{x}}), \ldots, \phi^n(\bar{\bar{x}})) = F^\alpha(\bar{\bar{x}}^1, \ldots, \bar{\bar{x}}^n);$$

here it is to be noticed that we write $T_b T_a$ as indicative of the fact that the transformation T_a is *first* applied and that this is followed by the transformation T_b. We have now satisfied the above Condition A. If we understand that a particular transformation (55.1) and

$$y^\alpha = f^\alpha(y^1, \ldots, y^n) \tag{55.2}$$

represent the same coordinate transformation T_a, we see immediately that Condition B is likewise satisfied. Evidently Condition C is satisfied by taking I as the identical transformation: $x^\alpha = \bar{x}^\alpha$. Now we observed in § 1 that the inverse of any transformation T_a necessarily exists and hence belongs to the set of all coordinate transformations of the region $\mathscr{R}$. Identifying this inverse with the element T_a^{-1} it is seen immediately that Condition D also holds. Hence *the set of all transformations of a region $\mathscr{R}$ constitutes a group*. An abstract group of this character is called *an infinite continuous group*.

Among the set of all continuous transformations of the coordinates of a region $\mathscr{R}$ we have the group of all continuous transformations possessing derivatives of orders 1 to p inclusive, and still more particularly the group of all analytic transformations of the coordinates of a region $\mathscr{R}$.

A group $\mathfrak{g}$ is said to be a *sub-group* of a group $\mathfrak{G}$ if (1) the elements of $\mathfrak{g}$ are contained among the elements of $\mathfrak{G}$ and (2) the elements of $\mathfrak{g}$ do not comprise the totality of elements of $\mathfrak{G}$. Thus the group of all analytic transformations of the coordinates of a region $\mathscr{R}$ is a sub-group of the group of all continuous transformations of the coordinates of this region.

56. Finite continuous groups

An abstract group is said to be *finite and continuous of order r* if its elements T_a generate a space of r ($\geqq 1$) dimensions, i.e. can be represented as the points of an r-dimensional space (see § 1); this space will be referred to in the following as the *group space* $\mathscr{A}$. It is to be assumed moreover that if two infinite sequences of elements T_{a_n} and T_{b_n} tend

respectively toward T_a and T_b, that the infinite sequence of elements $T_{a_n} T_{b_n}$ tends toward $T_a T_b$; also that if T_{a_n} tends toward I, $T_{a_n}^{-1}$ tends toward I. If V_0 is a neighbourhood of the group space $\mathfrak{A}$ containing the identity element I in its interior, the set of elements $T_a V_0$ obtained as the product of T_a and the elements of V_0 can be considered as another neighbourhood containing the element T_a in its interior; a similar remark can be made regarding the set of elements $V_0 T_a$.

We say that a finite and continuous abstract group is a *Lie group* if in a sufficiently small neighbourhood V_0 of the group space $\mathfrak{A}$, containing the identity element I, we can find a system of coordinates or parameters $a^1, \ldots, a^r$ such that the parameters c^α of the element $T_c = T_a T_b$ can be expressed in the form $c^\alpha = \phi^\alpha(a, b)$, where the functions ϕ^α admit continuous first and second derivatives. If the elements T_a of a Lie group represent point transformations in an n-dimensional space $\mathcal{T}$, these transformations can be defined by equations of the form

$$\bar{x}^\alpha = f^\alpha(x^1, \ldots, x^n;\ a^1, \ldots, a^r) \tag{56.1}$$

provided that the parameters are the coordinates of points lying in the above neighbourhood V_0. More precisely we can say that if $a^1, \ldots, a^r$ are the coordinates of a point of V_0 and the $x^1, \ldots, x^n$ are the coordinates of points P of an n-dimensional region $\mathcal{R}^*$ contained in the region $\mathcal{R}$ of $\mathcal{T}$ covered by the coordinate system, the points P of $\mathcal{R}^*$ will be transformed by (56.1) into points $\bar{P}$ lying in the region $\mathcal{R}$ (see Fig. 8). In fact it is evident that by suitably restricting the neighbourhood V_0, the resultant of any *finite* number of transformations (56.1), in which the parameters $a^1, \ldots, a^r$ are coordinates of points of V_0, will not carry the points of $\mathcal{R}^*$ outside of $\mathcal{R}$.

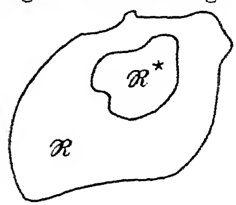

Fig. 8.

A special case of a Lie group of point transformations occurs when all the transformations of the group can be represented by (56.1); then the region $\mathcal{R}$ includes the entire space $\mathcal{T}$, and the above relations $c^\alpha = \phi^\alpha(a, b)$ apply to the entire group space $\mathfrak{A}$. Many important groups of point transformations are included under this special case. We shall in fact be concerned with transformation groups of this character in the applications of group theory to the discussion of scalar differential invariants and parameters in the present chapter; it will therefore be assumed in the following that the group under consideration can be represented completely by equations of the form (56.1).

The resultant of a particular transformation (56.1) and a second transformation of this set, namely

$$\bar{\bar{x}}^\alpha = f^\alpha(\bar{x}^1, \ldots, \bar{x}^n;\ b^1, \ldots, b^r), \tag{56.2}$$

where the b^α are coordinates of a point of the group space $\mathfrak{A}$, will be represented by

$$\bar{\bar{x}}^\alpha = f^\alpha(f^1(x, a), \ldots, f^n(x, a);\ b^1, \ldots, b^r); \tag{56.3}$$

this latter transformation must likewise transform any point P of the space $\mathcal{T}$ into a point $\bar{\bar{P}}$ of this space.

We shall suppose that the f^α in (56.1) are analytic functions of the variables $x^1, \ldots, x^n, a^1, \ldots, a^r$ when the x^α are coordinates of points of the space $\mathcal{T}$ and the a^α are coordinates of points of the group space $\mathfrak{A}$. As further conditions on these functions sufficient to insure that the equations (56.1) will define a Lie group of transformations of the space $\mathcal{T}$, we impose the following assumptions.

(*a*) *The functions* f^α *in* (56.3) *are such that*

$$f^\alpha(f^1(x, a), \ldots, f^n(x, a);\ b^1, \ldots, b^r) \equiv f^\alpha(x^1, \ldots, x^n;\ c^1, \ldots, c^r), \tag{56.4}$$

where the values $c^1, \ldots, c^r$ *are uniquely determined and are the coordinates of a point* C *contained in the group space* $\mathfrak{A}$. Hence if we denote any transformation (56.1) by the symbol T_a and if we denote the resultant transformation (56.3) of (56.1) and (56.2) by $T_b T_a$ as in the illustration of § 55, we see that the Conditions A and B of this latter section are satisfied.

(*b*) *The transformation inverse to any transformation* (56.1) *exists and belongs to the set*, i.e. this transformation is of the form

$$x^\alpha = f^\alpha(\bar{x}^1, \ldots, \bar{x}^n;\ \bar{a}^1, \ldots, \bar{a}^r), \tag{56.5}$$

where the values $\bar{a}^1, \ldots, \bar{a}^r$ are the coordinates of a point $\bar{A}$ contained in the group space $\mathcal{A}$. It follows from assumptions (*a*) and (*b*) that the identity transformation $\bar{x}^\alpha = x^\alpha$ exists and belongs to the set (56.1), i.e. that there exists a point A_0 in the group space $\mathcal{A}$ with coordinates $a_0^1, \ldots, a_0^r$ such that

$$f^\alpha(x^1, \ldots, x^n;\ a_0^1, \ldots, a_0^r) \equiv x^\alpha. \tag{56.6}$$

Identifying the element I with the identity transformation, Condition C of § 55 is satisfied. Finally when we take the element T_a^{-1} as the above transformation (56.5) we deduce the validity of the last Group Condition D.

We can likewise deduce from the above assumption (*b*) that the functional determinant

$$\left| \frac{\partial f^\alpha}{\partial x^\beta} \right| \neq 0 \tag{56.7}$$

for any points P and A in the spaces $\mathcal{X}$ and $\mathcal{A}$, respectively. From (56.1) and (56.5) we have the identity

$$\left| \frac{\partial x^\alpha}{\partial \bar{x}^\beta} \right| \left| \frac{\partial \bar{x}^\sigma}{\partial x^\tau} \right| = 1;$$

hence if one of the determinants in this identity were equal to zero, the other would become infinite contrary to the hypothesis of the analytic character of the functions f^α.

57. Essential parameters

Suppose that the number of parameters a^β in the set of transformations (56.1) can be decreased, i.e. that there exists $r-1$ functions $A^1, \ldots, A^{r-1}$ of the variables $a^1, \ldots, a^r$ such that the n equations

$$f^\alpha(x^1, \ldots, x^n;\ a^1, \ldots, a^r) \equiv F^\alpha(x^1, \ldots, x^n;\ A^1, \ldots, A^{r-1}) \tag{57.1}$$

hold for all values of the variables x^α and a^β. Since the matrix

$$\begin{matrix} \dfrac{\partial A^1}{\partial a^1} & \dfrac{\partial A^1}{\partial a^r} \\ & \\ \dfrac{\partial A^{r-1}}{\partial a^1} & \dfrac{\partial A^{r-1}}{\partial a^r} \end{matrix}$$

is at most of rank $r-1$, it follows that there exists r functions $\chi^1(a), \ldots, \chi^r(a)$ of the variables a^β, not all of which are identically zero, such that the functions $A^i(a)$ are given as solutions of the differential equations*

$$\sum_{k=1}^{r} \chi^k \frac{\partial f}{\partial a^k} = 0. \tag{57.2}$$

The above functions F^α must then also satisfy the equations (57.2) as well as the functions f^α whose equality with the functions F^α is expressed by (57.1).

Conversely, suppose that the functions f^α satisfy a system of the form (57.2); then the f^α can be expressed as functions of the variables $x^1, \ldots, x^n$ and $r-1$ functions $A^1, \ldots, A^{r-1}$ which are given as solutions of (57.2), i.e. an identity of the form (57.1) is satisfied.

If it is not possible to decrease the number of the parameters a^β in the functions f^α, i.e. if an identity of the form (57.1) does not exist, we say that the parameters a^β are *essential*. The following theorem can now be stated.

* Hereafter the summation sign will be omitted for indices which take on the values 1, ..., r as well as for those which take on the values 1, ..., n, in accordance with the usual convention.

THEOREM. *A necessary and sufficient condition that the r parameters a^β in the functions f^α in the set of transformations* (56.1) *be essential is that the f^α do not satisfy a system of equations of the form* (57.2).

It will be supposed in the following that the parameters $a^1, \ldots, a^r$ in the group represented by (56.1) are essential; the group (56.1) is then called an *r-parameter group*.

58. The parameter groups

Since the quantities c^α occurring in (56.4) are uniquely determined by hypothesis, we must have a definite relation

(58.1) $$c^\alpha = \phi^\alpha(a, b),$$

where the variables a^β and b^β are the coordinates of arbitrary points lying in the space $\mathfrak{A}$; evidently these functions ϕ^α are analytic in the variables b^β and a^β in consequence of the analyticity of the functions f^α in the transformation group (56.1). By the assumption (a) of § 56 the parameters c^α are also the coordinates of a point of the space $\mathfrak{A}$. Hence if we regard the b^β in (58.1) as a set of parameters, analogous to the parameters a^β in (56.1), the equations (58.1) will define an analytic point transformation of the space $\mathfrak{A}$ into itself. We shall now show that this set of point transformations (58.1) constitutes a group in the sense of § 55.

Let us first observe that for the values $b^\beta = a_0^\beta$ of the parameters in (58.1) we have from (56.4) and (56.5) that

$$f^\alpha(f(x, a), a_0) = f^\alpha(x, a) = f^\alpha(x, c).$$

Hence $c^\alpha = a^\alpha$ and equations (58.1) become

(58.2) $$c^\alpha = \phi^\alpha(a, a_0) \equiv a^\alpha.$$

Similarly if $a^\beta = a_0^\beta$ in (56.4) we have

$$f^\alpha(f(x, a_0), b) = f^\alpha(x, b) = f^\alpha(x, c),$$

i.e. $c^\beta = b^\beta$ so that (58.1) gives

(58.3) $$c^\alpha = \phi^\alpha(a_0, b) \equiv b^\alpha.$$

We have $$T_c = T_b T_a$$

as representative of the transformations of the group (56.1) from which the equations (58.1) were derived. Consider also the transformations

(58.4) $$T_q = T_p T_c = T_p(T_b T_a) = (T_p T_b) T_a = T_r T_a,$$

where

(58.5) $$T_r = T_p T_b.$$

Thus we see from (58.4) that

(58.6) $$q^\alpha = \phi^\alpha(c, p),$$

(58.7) $$q^\alpha = \phi^\alpha(a, r),$$

and from (58.5) that

(58.8) $$r^\alpha = \phi^\alpha(b, p).$$

In other words the resultant of the transformations (58.1) and (58.6) is the transformation (58.7) in which the parameters r^β are given by (58.8), i.e. the resultant of these two transformations belongs to the set (58.1).

To show that the transformation inverse to any transformation (58.1) exists and also belongs to this set, consider the transformation

$$T_a = T_s T_c$$

which defines T_s as a transformation of the set (56.1). This gives the parameter relation

(58.9) $$a^\alpha = \phi^\alpha(c, s)$$

as the inverse of the transformation (58.1); hence this inverse transformation belongs to the set (58.1).

Finally we see that the existence of the identical transformation (58.2) when combined with the above results shows that the complete set of group requirements A, ..., D of § 55 are satisfied by the set of transformations (58.1). Moreover on account of the identities (58.3), it is evident that the parameters b^β in (58.1) are essential, i.e. that the number of these parameters cannot be decreased by means of identities of the form (57.1). Hence *the set of transformations* (58.1) *considered as point transformations* $a \to c$ *with the parameters* b^β *constitute an r-parameter group, the identity transformation for this group being given by the values* a_0^β *which also yield the identity transformation of the original group* (56.1).

It can be shown in a similar manner that the set of point transformations (58.1) in which the a^β are regarded as the parameters likewise constitutes an r-parameter group for which the identity transformation is given by the values $a^\beta = a_0^\beta$ of the parameters; this group likewise defines an analytic point transformation of the space $\mathcal{A}$ upon itself. We shall accordingly refer to the group (58.1) with the b^β as parameters as the *first parameter group*, and to the group (58.1) for which the a^β are taken to be the parameters as the *second parameter group*.

Corresponding to the condition (56.7) we can now deduce the fact that the functional determinants

$$\left|\frac{\partial\phi^\alpha}{\partial a^\beta}\right| \neq 0, \quad \left|\frac{\partial\phi^\alpha}{\partial b^\beta}\right| \neq 0 \tag{58.10}$$

for the coordinates a^β and b^β of any two points A and B of the space $\mathcal{A}$.

59. Fundamental differential equations of an r-parameter group

Let us write the equations (56.4) in the form

$$\bar{\bar{x}}^\alpha = f^\alpha(\bar{x}, b) = f^\alpha(x, c), \tag{59.1}$$

and observe that these equations involve the quantities x^α, $\bar{x}^\alpha$, $\bar{\bar{x}}^\alpha$, b^α, c^α as well as the a^α since the functions $\bar{x}^\alpha$ depend on these latter parameters. Since the transformation (58.1) of the second parameter group can be written in the inverse form

$$b^\alpha = \zeta^\alpha(d, c), \tag{59.2}$$

where the parameters d^β depend on the a^β alone, we can consider the x^α, a^α, c^α as a set of independent variables and the $\bar{x}^\alpha$, $\bar{\bar{x}}^\alpha$, b^α as functions of these variables. Then, differentiating the above equations (59.1) with respect to the independent variables a^β, we have

$$\frac{\partial\bar{\bar{x}}^\alpha}{\partial\bar{x}^\sigma}\frac{\partial\bar{x}^\sigma}{\partial a^\beta} + \frac{\partial\bar{\bar{x}}^\alpha}{\partial b^k}\frac{\partial b^k}{\partial a^\beta} = 0.$$

Multiplying these latter equations through by $\partial\bar{x}^\gamma/\partial\bar{\bar{x}}^\alpha$ and summing on the index α, we obtain a set of equations of the form

$$\frac{\partial\bar{x}^\alpha}{\partial a^\beta} = -\frac{\partial\bar{x}^\alpha}{\partial\bar{\bar{x}}^\sigma}\frac{\partial\bar{\bar{x}}^\sigma}{\partial b^k}\frac{\partial b^k}{\partial a^\beta}. \tag{59.3}$$

Suppose that we now put b^β equal to the constants a_0^β which give the identity transformation of the group (56.1). Then the first set of derivatives $\partial\bar{x}^\alpha/\partial\bar{\bar{x}}^\sigma$ in the right members of (59.3) will be equal to the corresponding Kronecker δ_σ^α, and the second set of derivatives $\partial\bar{\bar{x}}^\sigma/\partial b^k$ will depend only on the $\bar{x}^\alpha$ as is seen directly from (59.1); also it follows directly from (59.2) that the last set of derivatives $\partial b^k/\partial a^\beta$ will depend on the variables a^β alone.

Hence, if we put

$$\frac{\partial b^k}{\partial a^\beta} = A_\beta^k(a), \tag{59.4}$$

the above equations (59.3) assume the form

$$\frac{\partial \bar{x}^\alpha}{\partial a^\beta} = A^k_\beta(a)\,\xi^\alpha_k(\bar{x}), \tag{59.5}$$

where the A^k_β and ξ^α_k are analytic functions of the coordinates a^α and $\bar{x}^\alpha$ of arbitrary points A and P of the spaces $\mathcal{A}$ and $\mathcal{T}$, respectively. The above equations (59.5) are of great importance in the study of the transformation group (56.1) and will consequently be called the *fundamental differential equations of the group* (56.1).

The determinant

$$|A^k_\beta(a)| \neq 0 \tag{59.6}$$

for the coordinates a^α of any point A of the space $\mathcal{A}$. To see this let us observe that if we differentiate (58.1) with respect to the independent variables a^β, we have

$$\frac{\partial c^\alpha}{\partial a^\beta} + \frac{\partial c^\alpha}{\partial b^k}\frac{\partial b^k}{\partial a^\beta} = 0,$$

and that these equations give, when multiplied by the derivatives $\partial b^\gamma/\partial c^\alpha$ obtained from the relations (59.2) inverse to (58.1), the set of equations

$$\frac{\partial b^k}{\partial a^\beta} = -\frac{\partial b^k}{\partial c^\sigma}\frac{\partial c^\sigma}{\partial a^\beta}. \tag{59.7}$$

The above condition (59.6) is a direct consequence of the equations (59.7) and the two conditions (58.10). Hence a set of quantities $A^\beta_k(a)$ are uniquely defined by the relations

$$A^i_\beta A^\beta_k = \delta^i_k, \quad A^k_\alpha A^\beta_k = \delta^\beta_\alpha,$$

where it is to be observed that these latter quantities $A^\beta_k(a)$ are distinguished from the quantities A^k_β by the use of Latin and Greek indices as in § 6. Multiplying the equations (59.5) by A^β_i and summing on the index β, these equations can therefore be solved so as to obtain

$$\xi^\alpha_k(\bar{x}) = A^\beta_k(a)\frac{\partial \bar{x}^\alpha}{\partial a^\beta}. \tag{59.8}$$

If (56.1) *represents an r-parameter group, then no set of constants $e^1, \ldots, e^r$, not all of which are zero, can be found such that the n linear homogeneous relations*

$$e^k \xi^\alpha_k(\bar{x}) \equiv 0 \tag{59.9}$$

are satisfied identically in the coordinates $\bar{x}^\beta$. To see this, we observe that if we could find a set of identities (59.9) we would have from (59.8) that

$$e^k A^\beta_k(a)\frac{\partial \bar{x}^\alpha}{\partial a^\beta} = 0,$$

i.e. a set of relations of the form (57.2) would exist contrary to the hypothesis that the parameters a^α in (56.1) are essential.

We can easily deduce the equations corresponding to (59.5) for the first and second parameter groups (58.1). Thus from the two sets of equations (59.1) we have

$$\frac{\partial \bar{\bar{x}}^\alpha}{\partial b^\beta} = A^k_\beta(b)\,\xi^\alpha_k(\bar{\bar{x}}), \tag{59.10}$$

$$\frac{\partial \bar{\bar{x}}^\alpha}{\partial c^\gamma} = A^k_\gamma(c)\,\xi^\alpha_k(\bar{\bar{x}}) \tag{59.11}$$

in consequence of (59.5). Now multiply the equations (59.11) by the derivatives $\partial c^\gamma/\partial b^\beta$ obtained by differentiation of (59.1) in which the a^α are held fixed. This gives

$$\frac{\partial \bar{\bar{x}}^\alpha}{\partial c^\gamma}\frac{\partial c^\gamma}{\partial b^\beta} = \frac{\partial \bar{\bar{x}}^\alpha}{\partial b^\beta} = A^k_\gamma(c)\,\xi^\alpha_k(\bar{\bar{x}})\frac{\partial c^\gamma}{\partial b^\beta}.$$

Hence from these latter equations and (59.10) we have

$$\left[A^k_\beta(b) - A^k_\gamma(c)\frac{\partial c^\gamma}{\partial b^\beta}\right]\xi^\alpha_k(\bar{\bar{x}}) = 0;$$

and this set of equations yields

$$\frac{\partial c^\alpha}{\partial b^\beta} = A_\beta^k(b)\, A_k^\alpha(c), \tag{59.12}$$

since otherwise we would have a set of equations of the form (59.9).

In a corresponding manner we now consider the two sets of transformation equations

$$x^\alpha = f^\alpha(\bar{x}, \bar{a}), \quad \bar{x}^\alpha = f^\alpha(\bar{\bar{x}}, b),$$

which are the inverses of (56.1) and (56.2) respectively. Then

$$x^\alpha = f^\alpha(\bar{x}, \bar{a}) = f^\alpha(f(\bar{\bar{x}}, \bar{b}), \bar{a}) = f^\alpha(\bar{\bar{x}}, \bar{c}), \tag{59.13}$$

where

$$\bar{c}^\alpha = \phi^\alpha(\bar{b}, \bar{a}). \tag{59.14}$$

Hence

$$\frac{\partial x^\alpha}{\partial \bar{a}^\beta} = A_\beta^k(\bar{a})\, \xi_k^\alpha(x), \tag{59.15}$$

$$\frac{\partial x^\alpha}{\partial \bar{c}^\gamma} = A_\gamma^k(\bar{c})\, \xi_k^\alpha(x) \tag{59.16}$$

from (59.5) and (59.13). Multiplying (59.16) by the derivatives $\partial\bar{c}^\gamma/\partial\bar{a}^\beta$ obtained by differentiation of (59.14), and then taking account of (59.15), we obtain a set of equations which can be reduced to the form

$$\frac{\partial c^\alpha}{\partial a^\beta} = A_\beta^k(a)\, A_k^\alpha(c), \tag{59.17}$$

where the bars over the letters have been omitted.

The equations (59.12) *and* (59.17) *are the fundamental differential equations for the first and second parameter groups respectively.*

60. Transformation theory connected with the fundamental differential equations

Suppose that $^\star A_\beta^k(a)$ and $^\star\xi_k^\alpha(\bar{x})$ is any other admissible set of the corresponding quantities in (59.5) so that we can write

$$\frac{\partial \bar{x}^\alpha}{\partial a^\beta} = {}^\star A_\beta^k(a)\, {}^\star\xi_k^\alpha(\bar{x}).$$

Then

$$^\star A_\beta^k(a)\, {}^\star\xi_k^\alpha(\bar{x}) = A_\beta^k(a)\, \xi_k^\alpha(\bar{x}), \tag{60.1}$$

and these equations must hold identically in the variables a^α and $\bar{x}^\alpha$. Multiplying (60.1) by $^\star A_i^\beta$, these equations assume the form

$$^\star\xi_k^\alpha(\bar{x}) = m_k^i\, \xi_i^\alpha(\bar{x}), \tag{60.2}$$

where

$$m_k^i = {}^\star A_k^\beta(a)\, A_\beta^i(a). \tag{60.3}$$

If we differentiate (60.2) with respect to the independent variables a^β, we obtain

$$\frac{\partial m_k^i}{\partial a^\beta}\, \xi_i^\alpha(\bar{x}) = 0;$$

hence the derivatives $\partial m_k^i/\partial a^\beta$ must vanish, since otherwise a relation of the form (59.9) would result, i.e. *the quantities m_k^i are constants.*

From the equations (60.3) we have

$$A_\alpha^k = m_i^k\, {}^\star A_\alpha^i. \tag{60.4}$$

Taking the determinants of both members of these equations, we see in consequence of a condition of the form (59.6) that the determinant $|m_k^i|$ cannot be equal to zero. Hence we can define constants $\tilde{m}_k^i$ so as to satisfy the equations

$$m_i^j \tilde{m}_k^i = \delta_k^j, \quad m_j^k \tilde{m}_k^i = \delta_j^i,$$

and the equations (60.4) can therefore be put into the form

(60.5) $$ {}^{*}A^i_\alpha = \bar{m}^i_k A^k_\alpha; $$

it also follows readily from (60.4) that the equations

(60.6) $$ {}^{*}A^\alpha_k = m^i_k A^\alpha_i $$

give the transformation of the quantities A^α_k. We can now state the following

THEOREM. *If the equations (59.5) are the fundamental differential equations of an r-parameter group, the quantities ξ^α_k and A^k_β in these equations are determined at most to within a transformation (60.2) and (60.5), in which the m^i_k are arbitrary constants subject to the condition that the determinant $|m^i_k|$ is not equal to zero.*

61. EQUIVALENT r-PARAMETER GROUPS

Consider an r-parameter group

(61.1) $$ \bar{y}^\alpha = F^\alpha(y^1, \ldots, y^n;\; b^1, \ldots, b^r), $$

with spaces $\mathcal{T}^*$ and $\mathcal{C}^*$ corresponding to the spaces $\mathcal{T}$ and $\mathcal{C}$, respectively, of the transformation group (56.1); thus (61.1) transforms the space $\mathcal{T}^*$ into itself and the functions F^α are analytic functions of the coordinates y^α and b^α of arbitrary points of the spaces $\mathcal{T}^*$ and $\mathcal{C}^*$, respectively. Suppose that the set of analytic relations

(61.2) $$ y^\alpha = \psi^\alpha(x^1, \ldots, x^n) $$

establishes a one to one reciprocal correspondence between the points of the space $\mathcal{T}^*$ and the space $\mathcal{T}$ of coordinates x^α; let us suppose that a similar correspondence is defined by the relations

(61.3) $$ b^\alpha = \omega^\alpha(a^1, \ldots, a^r) $$

between the points of the space $\mathcal{C}^*$ and points of the space $\mathcal{C}$. By this hypothesis the inverses of (61.2) and (61.3) exist, and will be represented by the equations

(61.4) $$ x^\alpha = \Psi^\alpha(y^1, \ldots, y^n), $$

and

(61.5) $$ a^\alpha = \Omega^\alpha(b^1, \ldots, b^r), $$

respectively. Hence the group (61.1) determines a point transformation of the space $\mathcal{T}$ upon itself. In fact we have

(61.6) $$ \bar{x}^\alpha = \Psi^\alpha(\bar{y}) = \Psi^\alpha(F(y,b)) = \Psi^\alpha(F(\psi(x), \omega(a))) = \bar{f}^\alpha(x^1, \ldots, x^n;\; a^1, \ldots, a^r), $$

where the $a^1, \ldots, a^r$ are the coordinates of an arbitrary point A of the space $\mathcal{C}$.

The above transformations, i.e.

(61.7) $$ \bar{x}^\alpha = \bar{f}^\alpha(x^1, \ldots, x^n;\; a^1, \ldots, a^r), $$

constitute a group. Let us first observe that the functions $\bar{f}^\alpha$ in these equations are analytic whenever the x^α and a^α are coordinates of points in the spaces $\mathcal{T}$ and $\mathcal{C}$, respectively. Moreover the parameters a^α in the functions $\bar{f}^\alpha$ are essential. If this were not the case we would have a set of relations of the form

$$ \chi^\beta(a)\frac{\partial \bar{f}^\alpha}{\partial a^\beta} = 0, $$

in which all of the χ^β did not vanish, and this would lead to the relations

$$ \left[\chi^\beta(a)\frac{\partial b^\tau}{\partial a^\beta}\right]\frac{\partial F^\sigma}{\partial b^\tau} = 0, $$

where not all of the above expressions in brackets would vanish; but this is contrary to the hypothesis that the parameters b^α in (61.1) are essential (see § 57). To show that the resultant of the transformations (61.7) and

(61.8) $$ \bar{\bar{x}}^\alpha = \bar{f}^\alpha(\bar{x}^1, \ldots, \bar{x}^n;\; p^1, \ldots, p^r) $$

belongs to the set (61.7), where the p^α are the coordinates of a point of the space $\mathfrak{A}$, we consider the transformation

$$\bar{y}^\alpha = F^\alpha(\bar{y}, c), \quad c^\alpha = \omega^\alpha(p).$$

Thus we have

$$\bar{\bar{x}}^\alpha = \Psi^{*\alpha}(\bar{\bar{y}}) = \Psi^{*\alpha}(F(\bar{y}, c)) = \bar{f}^\alpha(\bar{x}, p) \tag{61.9}$$
$$= \Psi^{*\alpha}(F(F(y, b), c)) = \Psi^{*\alpha}(F(y, d)) = \bar{f}^\alpha(x, q),$$

where the q^α are the coordinates of a point of the space $\mathfrak{A}$; that is, the transformation

$$\bar{\bar{x}}^\alpha = \bar{f}^\alpha(x, q),$$

which belongs to the set (61.7), is the resultant of the transformations (61.7) and (61.8). Also the identity transformation belongs to the set (61.7); in fact if b_0^α are the values of the parameters which give the identity transformation (61.1), we have from (61.6) that

$$\bar{x}^\alpha = \Psi^{*\alpha}(F(y, b_0)) = \Psi^{*\alpha}(y) = \Psi^{*\alpha}(\psi(x)) = x^\alpha;$$

that is

$$\bar{f}^\alpha(x^1, \ldots, x^n;\ a_0^1, \ldots, a_0^r) \equiv x^\alpha,$$

where the a_0^α are the values of the parameters a^α which correspond by (61.5) to the values b_0^α. Finally, the transformation inverse to any transformation (61.7) exists and belongs to the set (61.7). To show this we have merely to consider the transformation

$$y^\alpha = F^\alpha(\bar{y}, e),$$

inverse to the transformation (61.1), and to observe that

$$x^\alpha = \Psi^{*\alpha}(F(\bar{y}, e)) = \bar{f}^\alpha(x, r)$$

gives the transformation inverse to (61.7). Hence, *the transformations* (61.7) *are an r-parameter group of transformations of the space* $\mathfrak{X}$ *into itself, the parameters* a^α *being the coordinates of any point A of the space* $\mathfrak{A}$.

DEFINITION. *If the correspondences* (61.2) *and* (61.3) *can be chosen so that the relations*

$$f^\alpha(x^1, \ldots, x^n;\ a^1, \ldots, a^r) \equiv \bar{f}^\alpha(x^1, \ldots, x^n;\ a^1, \ldots, a^r)$$

are satisfied identically in the variables x^α *and* a^α, *the r-parameter groups* (56.1) *and* (61.1) *are said to be equivalent.*

When the group (61.1) is equivalent to the group (56.1) and the above correspondences (61.2) and (61.3) are known, the fundamental differential equations of the group (61.1) can readily be derived from the fundamental differential equations (59.5) of the group (56.1). We have

$$\frac{\partial \bar{y}^\alpha}{\partial b^\beta} = \frac{\partial \bar{y}^\alpha}{\partial \bar{x}^\sigma}\frac{\partial \bar{x}^\sigma}{\partial a^\tau}\frac{\partial a^\tau}{\partial b^\beta} = A_\tau^k(a)\frac{\partial a^\tau}{\partial b^\beta}\xi_k^\sigma(\bar{x})\frac{\partial \bar{y}^\alpha}{\partial \bar{x}^\sigma},$$

$$\frac{\partial \bar{y}^\alpha}{\partial b^\beta} = B_\beta^k(b)\,\zeta_k^\alpha(\bar{y}), \tag{61.10}$$

where

$$B_\beta^k(b) = A_\tau^k(a)\frac{\partial a^\tau}{\partial b^\beta}, \tag{61.11}$$

$$\zeta_k^\alpha(\bar{y}) = \xi_k^\sigma(\bar{x})\frac{\partial \bar{y}^\alpha}{\partial \bar{x}^\sigma}; \tag{61.12}$$

the equations (61.10) are then the fundamental differential equations of the group (61.1).

In view of the correspondences (61.2) and (61.3) there is no loss in generality in identifying the above spaces $\mathfrak{X}^*$ and $\mathfrak{A}^*$ for the group (61.1) with the spaces $\mathfrak{X}$ and $\mathfrak{A}$ for the group (56.1). Then the group (61.1), if equivalent to the group (56.1), can be thought of as derived from the latter by the coordinate transformations (61.2) and (61.3) of the spaces $\mathfrak{X}$ and $\mathfrak{A}$ respectively. On account of the equations (61.11) and (61.12) we can therefore say that *the quantities* $\xi_k^\alpha(x)$ *constitute the components of r contravariant vectors with respect to arbitrary analytic coordinate transformations* (61.2)

of the coordinates of the space $\mathcal{X}$, *and similarly the* A_α^k *are the components of r covariant vectors with respect to arbitrary analytic coordinate transformations* (61.3) *of the coordinates of the space* $\mathcal{A}$. It follows that the quantities A_k^α defined in § 59 are the components of r contravariant vectors of the space $\mathcal{A}$.

The vectors of the space $\mathcal{X}$ having the components ξ_k^α will be referred to as the *fundamental vectors of the r-parameter group* (56.1) since these components appear in the fundamental differential equations (59.5) of this group. Similarly the vectors of the space $\mathcal{A}$ with components A_k^α or A_α^k will be called the *fundamental vectors of the parameter groups* (58.1) as these components enter in an analogous manner into the fundamental differential equations of the parameter groups.

62. Constants of composition

We can readily deduce the system of equations

$$\left[A_p^\beta A_q^\gamma\left(\frac{\partial A_\beta^k}{\partial a^\gamma}-\frac{\partial A_\gamma^k}{\partial a^\beta}\right)\right]\xi_k^\alpha+\left(\xi_q^\sigma\frac{\partial \xi_p^\alpha}{\partial \bar{x}^\sigma}-\xi_p^\sigma\frac{\partial \xi_q^\alpha}{\partial \bar{x}^\sigma}\right)=0 \tag{62.1}$$

as the conditions of integrability of the fundamental differential equations (59.5), where the quantities A and ξ are functions of the variables a^α and $\bar{x}^\alpha$, respectively. Now since the variables a^α and $\bar{x}^\alpha$ can assume arbitrary values in the spaces $\mathcal{A}$ and $\mathcal{X}$, respectively, the equations (62.1) must be satisfied identically in these variables. Hence if we differentiate (62.1) with respect to a^σ, we see that the derivatives of the bracket expression in these equations must vanish identically since otherwise we would have a set of relations of the form (57.2); that is we must have

$$A_p^\beta A_q^\gamma\left(\frac{\partial A_\beta^k}{\partial a^\gamma}-\frac{\partial A_\gamma^k}{\partial a^\beta}\right)=C_{pq}^k, \tag{62.2}$$

where the C_{pq}^k are constants. Making the substitution (62.2) into (62.1) we then obtain

$$\xi_p^\sigma\frac{\partial \xi_q^\alpha}{\partial x^\sigma}-\xi_q^\sigma\frac{\partial \xi_p^\alpha}{\partial x^\sigma}=C_{pq}^k\,\xi_k^\alpha, \tag{62.3}$$

where, for simplicity, we have now omitted the bars over the coordinates x^α. The constants C_{pq}^k are called *the constants of composition of the r-parameter group* (56.1).

Let us now define the symbols X_pf by the identities

$$X_pf\equiv\xi_p^\alpha\frac{\partial f}{\partial x^\alpha},$$

in which the f denotes an arbitrary analytic function of the coordinates x^α. Let us also put

$$(X_pX_q)f=X_p(X_qf)-X_q(X_pf);$$

then when we expand the right members of these latter identities, we have

$$(X_pX_q)f=\left(\xi_p^\sigma\frac{\partial \xi_q^\alpha}{\partial x^\sigma}-\xi_q^\sigma\frac{\partial \xi_p^\alpha}{\partial x^\sigma}\right)\frac{\partial f}{\partial x^\alpha}.$$

Hence the equations (62.3) can be written

$$(X_pX_q)f=C_{pq}^kX_kf. \tag{62.4}$$

Correspondingly, if we put

$$Y_pf\equiv A_p^\alpha\frac{\partial f}{\partial a^\alpha},$$

where the f is now an arbitrary analytic function of the variables a^α, it is easily seen that the equations

$$(Y_pY_q)f=C_{pq}^kY_kf \tag{62.5}$$

are equivalent to the equations (62.2).

On account of (62.5) we see that the two parameter groups (58.1) have the same constants of composition as the group (56.1).

It follows immediately from (62.4) that the constants of composition C^k_{pq} satisfy the identities

$$C^k_{pq} = -C^k_{qp}. \tag{62.6}$$

Now the following identities

$$((X_p X_q) X_r) f + ((X_q X_r) X_p) f + ((X_r X_p) X_q) f = 0,$$

called the *Jacobi identities*, can be proved, for example, by direct computation; in consequence of these latter identities and (62.4) we immediately obtain

$$C^k_{pq} C^m_{kr} + C^k_{qr} C^m_{kp} + C^k_{rp} C^m_{kq} = 0 \tag{62.7}$$

as further conditions on the constants C^k_{pq}.

When the components ξ^α_k of the fundamental vectors of the r-parameter group (56.1) undergo the transformation (60.2), the constants of composition are transformed by the equations

$${}^*C^u_{vw} = C^k_{pq} \bar{m}^u_k m^p_v m^q_w;$$

this follows immediately from (60.5), (60.6) and (62.2).

63. Group space and its structure

Now we have seen that there exists in the group space $\mathfrak{A}$ a set of r independent contravariant vectors with components $A^\alpha_k(a)$, and that these components are determined to within a transformation of the form (60.6) in which the coefficients m^i_k are arbitrary constants subject to the condition that their determinant is not equal to zero. Hence, *the group space $\mathfrak{A}$ of an r-parameter group* (56.1) *is an r-dimensional space possessing the structure of a space of distant parallelism with the vectors of components A^α_k assuming the role of the fundamental vectors.* The group space $\mathfrak{A}$ is, however, a space of distant parallelism of restricted type since the components A^α_k of the fundamental vectors must satisfy the set of conditions (62.2).

The restriction on the structure of the group space $\mathfrak{A}$ imposed by the conditions (62.2) results in an interesting property of this space which we shall now deduce. Let us first observe that the equations (6.3) and (30.11) become

$$\Delta^\alpha_{\beta\gamma} = A^\alpha_i \frac{\partial A^i_\beta}{\partial a^\gamma},$$

$$\Lambda^\alpha_{\beta\gamma} = \tfrac{1}{2} A^\alpha_i \left(\frac{\partial A^i_\beta}{\partial a^\gamma} + \frac{\partial A^i_\gamma}{\partial a^\beta} \right),$$

respectively; also (35.13) becomes

$$h^i_{j,k} = \frac{1}{2} \left(\frac{\partial A^i_\alpha}{\partial a^\beta} - \frac{\partial A^i_\beta}{\partial a^\alpha} \right) A^\alpha_j A^\beta_k. \tag{63.1}$$

Hence (62.2) is equivalent to

$$h^i_{j,k} = \tfrac{1}{2} C^i_{jk}. \tag{63.2}$$

Making use of the conditions (63.2), the equations (48.3) reduce to

$$h^i_{j,kl} = \tfrac{1}{6} (C^i_{mk} C^m_{jl} + C^i_{ml} C^m_{jk}). \tag{63.3}$$

Hence, if we denote by $B^\alpha_{\beta\gamma\delta}$ the components of the curvature tensor B based on the components $\Lambda^\alpha_{\beta\gamma}$, and by B^i_{jkl} the components of this curvature tensor evaluated at the origin of a system of absolute coordinates, we have

$$B^i_{jkl} = \tfrac{1}{2} (h^i_{k,jl} - h^i_{l,jk}),$$

or

$$B^i_{jkl} = \tfrac{1}{4} C^m_{kl} C^i_{mj},$$

when use is made of (63.3) and the identities (62.6) and (62.7). Hence

$$B^{\alpha}_{\beta\gamma\delta}=\tfrac{1}{4}C^m_{kl}C^i_{mj}A^{\alpha}_iA^j_{\beta}A^k_{\gamma}A^l_{\delta}, \tag{63.4}$$

and

$$B_{\beta\gamma}\equiv B^{\alpha}_{\beta\gamma\alpha}=\tfrac{1}{4}g_{\beta\gamma}, \tag{63.5}$$

where we have put

$$g_{\beta\gamma}=C^p_{qr}C^q_{sp}A^r_{\beta}A^s_{\gamma}. \tag{63.6}$$

We can therefore introduce into the group space a fundamental metric tensor with components $g_{\alpha\beta}$ defined by (63.6) provided that the determinant $|g_{\alpha\beta}|$ formed from these components does not vanish at any point of this space. Since the determinant $|A^{\alpha}_i|$ does not vanish at any point A of the group space $\mathcal{A}$, the condition that the determinant $|g_{\alpha\beta}|$ should not vanish at any point of the space $\mathcal{A}$ is that the determinant

$$|C^p_{qr}C^q_{sp}| \tag{63.7}$$

is not equal to zero.

Now transform (63.4) to a system of absolute normal coordinates, differentiate and evaluate at the origin of this system. By making use of (63.2) and the identities (62.6) and (62.7) it can readily be deduced that the expressions, which are thereby obtained from the right members of (63.4), vanish identically. In other words the covariant derivative, with respect to the components $\Lambda^{\alpha}_{\beta\gamma}$, of the curvature tensor B, vanishes, and we therefore obtain from (63.5) that

$$g_{\alpha\beta,\gamma}\equiv\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}-g_{\sigma\beta}\Lambda^{\sigma}_{\alpha\gamma}-g_{\alpha\sigma}\Lambda^{\sigma}_{\beta\gamma}\equiv 0. \tag{63.8}$$

Hence, *if the determinant* (63.7) *is not equal to zero, we can introduce a metric into the group space $\mathcal{A}$ such that this space will become an Einstein space* of the special type for which the covariant derivative of the curvature tensor vanishes.*

64. Infinitesimal transformations

Consider the system of total differential equations

$$\frac{da^{\beta}}{dt}=\gamma^k A^{\beta}_k(a), \tag{64.1}$$

in which the γ^k are arbitrary constants; these equations have a unique solution $a^{\beta}(t)$ satisfying the set of initial conditions $a^{\beta}=a^{\beta}_0$ for $t=0$, where the a^{β}_0 are the values of the parameters which give the identity transformation (56.1). In fact if we put $u^k=\gamma^k t$, the quantities a^{β} are given by a power series in the variables u^k which will converge for sufficiently small values of the u^k. Thus we have

$$a^{\beta}=a^{\beta}_0+A^{\beta}_k(a_0)u^k+\tfrac{1}{2}A^{\gamma}_k(a_0)\frac{\partial A^{\beta}_l(a_0)}{\partial a^{\gamma}_0}u^k u^l+\dots. \tag{64.2}$$

For $u^k=0$, the determinant $|\partial a^{\beta}/\partial u^k|$ is equal to the determinant $|A^{\beta}_k(a_0)|$ which we saw in § 59 to be different from zero. The transformation inverse to (64.2) therefore exists and hence these equations can be regarded as defining a transformation of the parameters $a^{\beta}\rightarrow u^k$ within a sufficiently small domain $\mathcal{N}$ containing the point A_0 of the group space $\mathcal{A}$ which corresponds to the identity transformation (56.1). These new parameters u^k will be called *canonical parameters*; they are evidently analogous to the absolute normal coordinates of Chapter V.

If we multiply both members of the fundamental differential equations (59.5) of the

* That is, a space in which the equations

$$B_{\beta\gamma}=\lambda g_{\beta\gamma}$$

are satisfied, where λ is a constant.

group (56.1) by the corresponding members of (64.1), we obtain a set of equations which can be written

$$\frac{d\bar{x}^\alpha}{dt} = \gamma^k \bar{X}_k \bar{x}^\alpha, \tag{64.3}$$

where we have put

$$\bar{X}_k f = \xi_k^\alpha(\bar{x}) \frac{\partial f(\bar{x})}{\partial \bar{x}^\alpha}$$

in conformity with the notation of § 62. Suppose, now, that (56.1) becomes

$$\bar{x}^\alpha = F^\alpha(x^1, \ldots, x^n;\ u^1, \ldots, u^r) \tag{64.4}$$

as the result of the substitution (64.2). Then it is evident that (64.4), in which $u^k = \gamma^k t$, must satisfy (64.3) and hence the power series expansions of the right members of (64.4) can be obtained as the solution of (64.3) determined by the initial conditions $\bar{x}^\alpha = x^\alpha$, $t = 0$. We thus obtain the equations

$$\bar{x}^\alpha = x^\alpha + (X_k x^\alpha)\, u^k + \frac{1}{2!}(X_k X_l x^\alpha)\, u^k u^l + \frac{1}{3!}(X_k X_l X_m x^\alpha)\, u^k u^l u^m + \ldots \tag{64.5}$$

equivalent to (64.4).

Now the equations (64.5) define a transformation of the group (56.1) for sufficiently small values of the variables u^k. The equations (64.5) are sometimes referred to as the finite equations of transformation of the group; however, the equations (64.5) do not in general define all transformations of the group (56.1), but only those in the neighbourhood of the identity transformation. From (64.5) we derive the equations

$$\bar{x}^\alpha = x^\alpha + (X_k x^\alpha)\, u^k$$

by neglecting terms of higher order in the u^k. These latter transformations are called the infinitesimal transformations of the group although they do not in general give a transformation of the group (56.1). Since the infinitesimal transformations and therefore also the series (64.5) are determined when the symbols $X_k f$ are known, it is customary to refer to the $X_k f$ as *the symbols of the infinitesimal transformations of the group* (56.1).

Any transformation of the group (56.1) *can be generated by the successive application of a finite number of transformations* (64.5). Consider the transformation (56.1) determined by a point A_1 with coordinates a_1^α in the group space $\mathfrak{A}$; join the point A_0 which determines the identity transformation (56.6) to the point A_1 by a continuous curve C, which can be done since the space $\mathfrak{A}$ is connected (see § 1). Let da^α determine an infinitesimal displacement along C, e.g. a displacement from a point A with coordinates a^α to a point B with coordinates $a^\alpha + da^\alpha$, the points A and B lying on the curve C. The transformation corresponding to this displacement da^α will be represented by $T_{a+da} T_a^{-1}$ and so will belong to the group (56.1). But $T_{a+da} T_a^{-1}$ is a transformation in the neighbourhood of the identity and hence is determined by a set of parameters a^β defined by (64.2) for suitable values of the constants γ^k. The transformation $T_{a+da} T_a^{-1}$ can therefore be represented by (64.5). It is evident therefore that we can pass along the curve C from A_0 to A_1 by a finite number of displacements da^α, or in other words that the transformation (56.1) determined by the point A_1 can be broken up into a finite number of infinitesimal displacements (64.5).*

The curves in the space $\mathfrak{X}$ defined by (64.5), in which the γ^k are constants and t enters as a parameter, are called the *path curves* of the group (56.1). The above result can then be stated by saying that any transformation of the group (56.1) can be generated by displacements of the points of the space $\mathfrak{X}$ along suitable path curves.

* In fact if we could not proceed from A_0 to A_1 by a finite number of displacements da^α, the sequence of points of C determined by the displacements da^α would have a limit point L which might possibly coincide with the point A_1; but this is impossible since L could be taken as the point A and an associated displacement da^α would then carry us to a point of C on either side of the point L.

65. Transitive and intransitive groups. Invariant sub-spaces

Suppose that an arbitrary point P of the space $\mathcal{T}$ is subjected to all the transformations of the group (56.1). As a result of these transformations the point P will be displaced into the points of a surface S lying in the space $\mathcal{T}$, the surface S being defined by the parametric equations (56.1) in which the x_α are the coordinates of the point P; evidently the point P will lie on this surface S since (56.1) contains the identity transformation.

Now it is obvious that any point Q on the surface S can be displaced into any other point R of this surface by a suitable transformation of the group (56.1); on this account the surface S is called an *invariant* sub-space of the space $\mathcal{T}$. Since the point P can be displaced into any other point of S by a transformation (56.1), the surface S is said to be the *minimum invariant sub-space associated with the point* P.*

If we consider the transformations (64.5) in which $u^k = \gamma^k t$, and the x^α are the coordinates of the point P, we see that this point is displaced by transformations of the group (56.1) along directions defined by

$$\frac{d\bar{x}^\alpha}{dt} = \gamma^k \, \xi_k^\alpha(x), \tag{65.1}$$

where the γ^k are arbitrary constants. If the above surface S is a space of q dimensions there must be q independent directions in the set (65.1). Hence, *if the minimum invariant sub-space S associated with the point P is a space of q dimensions, the matrix*

$$\begin{matrix} \xi_1^1(x) & & \xi_1^n(x) \\ & & \\ \xi_r^1(x) & \cdots & \xi_r^n(x) \end{matrix} \tag{65.2}$$

is of rank q. Conversely, if this matrix is of rank q the minimum invariant sub-space associated with the point P is a space of q dimensions. The converse part of this statement is of course immediately evident.

The dimensionality q of the minimum invariant sub-space S associated with the point P can have any value from 0 to n inclusive. If $q=0$, the point P is invariant under all transformations of the group (56.1). When $q=n$, the minimum invariant sub-space S associated with the point P is the space $\mathcal{T}$ itself.†

If for a point P of the space $\mathcal{T}$ the minimum invariant sub-space S is the space $\mathcal{T}$ itself, the same will be true of any other point of $\mathcal{T}$, and the group (56.1) will be said to be *transitive*; otherwise the group is called *intransitive*. Hence a necessary condition for the group (56.1) to be transitive is that $r \geqq n$. A transitive group (56.1) for which $r=n$ is said to be *simply transitive*, otherwise it is called *multiply transitive*.‡

* It is clear that if the point P lies on an invariant surface, this surface is not necessarily the minimum invariant sub-space for the point P. For example, P may be displaced only along a curve C lying on the above surface by all transformations of the group (56.1); in this case the curve C will be the minimum sub-space associated with P.

† If $q=n$ and the minimum invariant sub-space S was an n-dimensional region $\mathcal{R}$ contained in $\mathcal{T}$, then points of $\mathcal{R}$ would be transformed into points of this same region by all transformations of the group (56.1); in this case we could replace the space $\mathcal{T}$ by the region $\mathcal{R}$ so as to secure the above condition. The region $\mathcal{R}^*$ composed of all points of $\mathcal{T}$ not contained in $\mathcal{R}$ constitutes a second region of transformation of the group (56.1). Evidently such a decomposition of the space $\mathcal{T}$ into the two regions $\mathcal{R}$ and $\mathcal{R}^*$ which is hereby assumed involves no loss of generality since each of these regions can be treated separately.

‡ The first and second parameter groups (58.1) are transitive groups. When the transformations (56.1) are confined to a minimum invariant sub-space S, they evidently define a group; this group is, of course, transitive and is called *the group induced on the minimum invariant sub-space S.*

66. Invariant functions

Consider an analytic function $f(x)$ of the variables x^α defined throughout the space $\mathscr{T}$. In general the function $f(\bar{x})$ will depend on the variables x^α and the parameters a^α as a result of the substitution (56.1). Now observe that

$$\frac{\partial f(\bar{x})}{\partial a^\beta} = \frac{\partial f(\bar{x})}{\partial \bar{x}^\alpha}\frac{\partial \bar{x}^\alpha}{\partial a^\beta} = A^k_\beta\, \xi^\alpha_k(\bar{x})\frac{\partial f(\bar{x})}{\partial \bar{x}^\alpha}$$

on account of (59.5). Hence the necessary and sufficient condition that $f(\bar{x})$ shall be independent of the parameters a^α is that

$$X_k f(x) = 0. \tag{66.1}$$

If the equations (66.1) are satisfied identically in the variables x^α, we therefore have that the function $f(\bar{x})$ becomes a function $F(x)$ as a result of the substitution (56.1). But $f(\bar{x}) = f(x)$ for the identity transformation (56.6); hence the functions $f(x)$ and $F(x)$ must be identically equal throughout the space $\mathscr{T}$ if the conditions (66.1) are satisfied.

A function $f(x)$ defined in $\mathscr{T}$ which remains unaltered in form under all transformations of the group (56.1) is said to be *invariant* under these transformations or to *admit the transformations of the group* (56.1). This gives us the following

Theorem. *A necessary and sufficient condition that an analytic function $f(x)$ defined throughout the space $\mathscr{T}$ admit the transformations of the group* (56.1) *is that the equations* (66.1) *be satisfied identically.*

It is evident that if the group (56.1) is transitive, it can admit no invariant function $f(x)$; in fact if such a function existed, the equation $f = \text{const.}$ would define an invariant sub-space S in $\mathscr{T}$ contrary to the hypothesis that the group is transitive.

Now the linear homogeneous differential equations (66.1) considered as equations for the determination of the function f constitute a complete set on account of the existence of (62.4). If there are m independent equations in the set (66.1) there will be $n-m$ functionally independent analytic solutions $f_1, \ldots, f_{n-m}$, each of which can be taken to be defined throughout a sufficiently small domain $\mathscr{S}$ of the space $\mathscr{T}$; any solution f of (66.1) analytic in $\mathscr{S}$ can then be expressed by a suitably chosen function of the solutions $f_1, \ldots, f_{n-m}$, which are said to form a fundamental system of solutions of (66.1). Now we can assume without real loss of generality that the points of the above domain $\mathscr{S}$ are transformed into all points of the space $\mathscr{T}$ by the transformations (56.1).* By treating each of the functions $f_1, \ldots, f_{n-m}$ as we did the function f in the derivation of the above theorem, we see therefore that these functions must actually be defined throughout the space $\mathscr{T}$ and that moreover they are unaltered in form by transformations of the group (56.1). We summarize these results in the following statement.

If there are $m\,(<n)$ independent equations (66.1) *there exist $n-m$ functionally independent functions $f_1, \ldots, f_{n-m}$, given as solutions of the equations* (66.1), *which admit the transformations of the group* (56.1). *Any analytic function defined throughout $\mathscr{T}$ which admits the transformations* (56.1) *can be expressed as a suitable function of the $f_1, \ldots, f_{n-m}$, and conversely any analytic function of $f_1, \ldots, f_{n-m}$ defined throughout $\mathscr{T}$ admits the transformations of the group* (56.1).

When there are $n-m$ mutually independent functions admitting the transformations of the group, the minimum invariant sub-space S associated with any point P of the space $\mathscr{T}$ is evidently defined by a system of equations of the form

$$f_1 = \text{const.}, \ldots, f_{n-m} = \text{const.}$$

* In fact, by the transformations (56.1) the domain $\mathscr{S}$ will go into an n-dimensional domain Δ of $\mathscr{T}$ which is composed of all the minimum invariant sub-spaces associated with the points of $\mathscr{S}$. If Δ does not include the entire space $\mathscr{T}$, we can replace $\mathscr{T}$ by Δ in the original definition of the group (56.1). The region $\mathscr{R}$ composed of all points of $\mathscr{T}$ not contained in Δ will then form a new ... (56.1) which can be treated separately.

In general the invariant sub-spaces will be spaces of m dimensions although for exceptional points the minimum invariant sub-spaces can be of smaller dimensionality.

67. Groups defined by the equations of transformation of the components of tensors

Consider the equations of transformation of the components of a relative tensor T of weight W, namely

$$\bar{T}^{\mu\ldots\nu}_{\eta\ldots\zeta} = |u^a_b|^W T^{\alpha\ldots\beta}_{\gamma\ldots\delta} u^\gamma_\eta \ldots u^\delta_\zeta \bar{u}^\mu_\alpha \ldots \bar{u}^\nu_\beta, \tag{67.1}$$

where

$$u^\gamma_\eta = \frac{\partial x^\gamma}{\partial \bar{x}^\eta}, \quad \bar{u}^\mu_\alpha = \frac{\partial \bar{x}^\mu}{\partial x^\alpha}.$$

If we do not consider the u^α_β as derivatives but as arbitrary parameters, then it is evident that (67.1) defines a group in the independent variables $T^{\alpha\ldots\beta}_{\gamma\ldots\delta}$ and the n^2 parameters u^α_β, provided that the determinant

$$|u^\alpha_\beta| \neq 0, \tag{67.2}$$

and the $\bar{u}^\mu_\alpha$ are determined in terms of the u^α_β by the relations

$$u^\alpha_\beta \bar{u}^\mu_\alpha = \delta^\mu_\beta, \quad u^\beta_\mu \bar{u}^\mu_\alpha = \delta^\beta_\alpha.$$

Likewise any number of equations of the type (67.1) will form a system by which a group is defined in a similar manner. The group space $\mathcal{A}$ of such a group can be taken as an n^2-dimensional Euclidean space, referred to coordinates u^α_β, with the exclusion of the hypersurface defined by $|u^\alpha_\beta| = 0$; taking the independent variables T as the coordinates of a Euclidean space $\mathcal{T}$, the group will then represent transformations of the space $\mathcal{T}$ upon itself.

It follows directly from the replacement theorem (see § 39) that any affine differential invariant of order p ($\geqq 1$) can be obtained from the group

$$\tag{67.3} \begin{cases} \bar{A}^\alpha_{\beta\gamma\delta} = A^\mu_{\nu\eta\zeta} u^\nu_\beta u^\eta_\gamma u^\zeta_\delta \bar{u}^\alpha_\mu, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \bar{A}^\alpha_{\beta\gamma\delta_1\ldots\delta_p} = A^\mu_{\nu\eta\zeta_1\ldots\zeta_p} u^\nu_\beta \ldots u^{\zeta_p}_{\delta_p} \bar{u}^\alpha_\mu, \end{cases}$$

by a suitable elimination of the parameters u^α_β. In a corresponding manner, any metric differential invariant of order p ($\geqq 0$) is obtainable from the group defined by

$$\tag{67.4} \begin{cases} \bar{g}_{\alpha\beta} = g_{\mu\nu} u^\mu_\alpha u^\nu_\beta, \\ \bar{g}_{\alpha\beta,\gamma_1\gamma_2} = g_{\mu\nu,\eta_1\eta_2} u^\mu_\alpha \ldots u^{\eta_2}_{\gamma_2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \bar{g}_{\alpha\beta,\gamma_1\ldots\gamma_p} = g_{\mu\nu,\eta_1\ldots\eta_p} u^\mu_\alpha \ldots u^{\eta_p}_{\gamma_p}; \end{cases}$$

here it is to be observed that the case $p=0$ corresponds to the first set of equations (67.4) and that the case $p=1$ does not arise.

When we speak of the group (67.3) we shall understand that any two variables $A^{\alpha}_{\beta_1\ldots\beta_m}$ and $A^{\mu}_{\gamma_1\ldots\gamma_m}$ are distinct if not all the equations

$$\alpha=\mu,\ \beta_1=\gamma_1,\ \ldots,\ \beta_m=\gamma_m$$

are satisfied simultaneously; obviously two variables $A^{\alpha}_{\beta_1\ldots\beta_m}$ and $A^{\alpha}_{\gamma_1\ldots\gamma_r}$ for which $m\neq r$ are to be considered as distinct. No attention is therefore to be paid *at present* to the complete set of identities (41.1) and (41.2) in the determination of the independent variables A; thus the variables A^1_{112} and A^1_{121} are distinct independent variables A, and A^1_{111} is to be considered as one of the independent variables A even though this quantity vanishes by (41.7). Similar remarks apply to the group (67.4). The groups (67.3) and (67.4) will be referred to as the *affine* and *metric groups*, respectively.

68. Infinitesimal transformations of the affine and metric groups

The n^2 parameters u^{α}_{β} in the affine or metric groups are *essential* regardless of the value of p. For example, let us consider the affine group for $p=1$, i.e.

$$\bar{A}^{\alpha}_{\beta\gamma\delta}=A^{\mu}_{\nu\eta\zeta}u^{\nu}_{\beta}u^{\eta}_{\gamma}u^{\zeta}_{\delta}\bar{u}^{\alpha}_{\mu}. \tag{68.1}$$

Differentiating (68.1) with respect to u^{σ}_{τ} and eliminating the variables $A^{\mu}_{\nu\eta\zeta}$ in the right members of the resulting equations by means of the equations (68.1) themselves, we obtain

$$\frac{\partial\bar{A}^{\alpha}_{\beta\gamma\delta}}{\partial u^{\sigma}_{\tau}}=\overline{\begin{pmatrix}\alpha\mu\\ \beta\gamma\delta\nu\end{pmatrix}}\,\bar{u}^{\nu}_{\sigma}\delta^{\tau}_{\mu}, \tag{68.2}$$

where

$$\begin{pmatrix}\alpha\mu\\ \beta\gamma\delta\nu\end{pmatrix}=A^{\alpha}_{\nu\gamma\delta}\delta^{\mu}_{\beta}+A^{\alpha}_{\beta\nu\delta}\delta^{\mu}_{\gamma}+A^{\alpha}_{\beta\gamma\nu}\delta^{\mu}_{\delta}-A^{\mu}_{\beta\gamma\delta}\delta^{\alpha}_{\nu}; \tag{68.3}$$

the bar over the parenthesis expression in (68.2) denotes the corresponding quantities in the variables $\bar{A}$. The condition that the n^2 parameters u^{α}_{β} in (68.1) be essential is that it be impossible to satisfy the equations

$$\chi^{\sigma}_{\tau}\frac{\partial\bar{A}^{\alpha}_{\beta\gamma\delta}}{\partial u^{\sigma}_{\tau}}=0,$$

by quantities χ^{σ}_{τ} not all zero, which are functions of the parameters alone (see § 57). This gives

$$\overline{\begin{pmatrix}\alpha\mu\\ \beta\gamma\delta\nu\end{pmatrix}}\,{}^{\star}\chi^{\nu}_{\mu}=0, \tag{68.4}$$

where ${}^{\star}\chi^{\nu}_{\mu}=\chi^{\sigma}_{\mu}\bar{u}^{\nu}_{\sigma}$. Differentiating (68.4) with respect to $\bar{A}^{\rho}_{\epsilon\zeta\eta}$, we obtain

$$[\delta^{\alpha}_{\rho}\delta^{\epsilon}_{\nu}\delta^{\zeta}_{\gamma}\delta^{\eta}_{\delta}\delta^{\mu}_{\beta}+\delta^{\alpha}_{\rho}\delta^{\epsilon}_{\beta}\delta^{\zeta}_{\nu}\delta^{\eta}_{\delta}\delta^{\mu}_{\gamma}+\delta^{\alpha}_{\rho}\delta^{\epsilon}_{\beta}\delta^{\zeta}_{\gamma}\delta^{\eta}_{\nu}\delta^{\mu}_{\delta}-\delta^{\mu}_{\rho}\delta^{\epsilon}_{\beta}\delta^{\zeta}_{\gamma}\delta^{\eta}_{\delta}\delta^{\alpha}_{\nu}]\,{}^{\star}\chi^{\nu}_{\mu}=0,$$

or

$${}^{\star}\chi^{\epsilon}_{\beta}\delta^{\alpha}_{\rho}\delta^{\zeta}_{\gamma}\delta^{\eta}_{\delta}+{}^{\star}\chi^{\zeta}_{\gamma}\delta^{\alpha}_{\rho}\delta^{\epsilon}_{\beta}\delta^{\eta}_{\delta}+{}^{\star}\chi^{\eta}_{\delta}\delta^{\alpha}_{\rho}\delta^{\epsilon}_{\beta}\delta^{\zeta}_{\gamma}-{}^{\star}\chi^{\alpha}_{\rho}\delta^{\epsilon}_{\beta}\delta^{\zeta}_{\gamma}\delta^{\eta}_{\delta}=0.$$

Putting $\alpha=\rho$, $\gamma=\zeta$, $\delta=\eta$ in these latter equations and summing on the repeated indices, gives

$$n\,{}^*\chi^\epsilon_\beta+{}^*\chi^\sigma_\sigma\delta^\epsilon_\beta=0.$$

When we take $\epsilon=\beta$ and sum on the repeated indices, the above equations show that ${}^*\chi^\sigma_\sigma=0$; hence ${}^*\chi^\epsilon_\beta=0$ and it follows that $\chi^\alpha_\beta=0$ in consequence of (67.2). This proves the essential character of the parameters in (68.1).

The proof that the parameters u^α_β in (67.4) for $p=0$, i.e. in the group

$$\bar{g}_{\alpha\beta}=g_{\mu\nu}u^\mu_\alpha u^\nu_\beta, \tag{68.5}$$

are essential, can be made in an analogous manner. Thus we have

$$\frac{\partial\bar{g}_{\alpha\beta}}{\partial u^\sigma_\tau}=\begin{bmatrix}\mu\\ \alpha\beta\nu\end{bmatrix}\bar{u}^\mu_\sigma\delta^\tau_\mu, \tag{68.6}$$

where

$$\begin{bmatrix}\mu\\ \alpha\beta\nu\end{bmatrix}=g_{\nu\beta}\delta^\mu_\alpha+g_{\alpha\nu}\delta^\mu_\beta. \tag{68.7}$$

Corresponding to (68.4) we now obtain

$$\begin{bmatrix}\mu\\ \alpha\beta\nu\end{bmatrix}{}^*\chi^\nu_\mu=0; \tag{68.8}$$

hence

$${}^*\chi^\epsilon_\alpha\delta^\zeta_\beta+{}^*\chi^\zeta_\beta\delta^\epsilon_\alpha=0.$$

When we now put $\alpha=\zeta$ and sum on the repeated indices we obtain ${}^*\chi^\epsilon_\beta=0$, which proves that the n^2 parameters u^α_β in the group (68.5) are essential.

It follows therefore that the n^2 parameters u^α_β are essential in the groups (67.3) and (67.4) for any value of p for which these groups are defined.

The equations (68.2) and (68.6) now show immediately that

$$X^\mu_\nu(1)f\equiv\begin{pmatrix}\alpha\mu\\ \beta\gamma\delta\nu\end{pmatrix}\frac{\partial f}{\partial A^\alpha_{\beta\gamma\delta}} \tag{68.9}$$

and

$$Y^\mu_\nu(0)f\equiv\begin{bmatrix}\mu\\ \alpha\beta\nu\end{bmatrix}\frac{\partial f}{\partial g_{\alpha\beta}} \tag{68.10}$$

are the symbols of the infinitesimal transformations of the groups (68.1) and (68.5), respectively (see § 64); we observe that repeated indices in these equations are to be summed and that the notation in their left members indicates the value of p in the groups (67.3) and (67.4) to which the infinitesimal transformations correspond. Similarly the infinitesimal transformations of the group (67.4) for $p=2$ are given by

$$Y^\mu_\nu(2)f\equiv\begin{bmatrix}\mu\\ \alpha\beta\nu\end{bmatrix}\frac{\partial f}{\partial g_{\alpha\beta}}+\begin{bmatrix}\mu\\ \alpha\beta\gamma\delta\nu\end{bmatrix}\frac{\partial f}{\partial g_{\alpha\beta,\gamma\delta}} \tag{68.11}$$

where

$$\begin{bmatrix}\mu\\ \alpha\beta\gamma\delta\nu\end{bmatrix}=g_{\nu\beta,\gamma\delta}\delta^\mu_\alpha+\dots+g_{\alpha\beta,\gamma\nu}\delta^\mu_\delta. \tag{68.12}$$

For an arbitrary value of p the symbol of the infinitesimal transformations of the group (67.3) is

$$X^\mu_\nu(p)f = \sum_{q=1}^{p} \begin{pmatrix} \alpha\mu \\ \beta\gamma\delta_1 \ldots \delta_q\nu \end{pmatrix} \frac{\partial f}{\partial A^\alpha_{\beta\gamma\delta_1\ldots\delta}} \tag{68.13}$$

where

$$\begin{pmatrix} \alpha\mu \\ \beta\gamma\delta_1 \ldots \delta_q\nu \end{pmatrix} = A^\alpha_{\nu\gamma\delta_1\ldots\delta_q}\delta^\mu_\beta + \ldots + A^\alpha_{\beta\gamma\delta_1\ldots\nu}\delta^\mu_{\delta_q} - A^\mu_{\beta\gamma\delta_1\ldots\delta_q}\delta^\alpha_\nu. \tag{68.14}$$

Also the symbol of the infinitesimal transformations of the general group (67.4) is defined by

$$Y^\mu_\nu(p)f = \begin{bmatrix} \mu \\ \alpha\beta\nu \end{bmatrix} \frac{\partial f}{\partial g_{\alpha\beta}} + \sum_{q=2}^{p} \begin{bmatrix} \mu \\ \alpha\beta\gamma_1 \ldots \gamma_q\nu \end{bmatrix} \frac{\partial f}{\partial g_{\alpha\beta,\gamma_1\ldots\gamma_q}}, \tag{68.15}$$

where

$$\begin{bmatrix} \mu \\ \alpha\beta\gamma_1 \ldots \gamma_q\nu \end{bmatrix} = g_{\nu\beta,\gamma_1\ldots\gamma_q}\delta^\mu_\alpha + \ldots + g_{\alpha\beta,\gamma_1\ldots\nu}\delta^\mu_{\gamma_q}. \tag{68.16}$$

The determination of (68.13) and (68.15) is of course completely analogous to the determination of the infinitesimal transformations (68.9) and (68.10) which has been given in detail.

69. Differential equations of absolute affine and metric scalar differential invariants

By the results of § 66 we have the following

Theorem. *A necessary and sufficient condition that the functions*

$$G(A^\alpha_{\beta\gamma\delta_1};\ \ldots;\ A^\alpha_{\beta\gamma\delta_1\ldots\delta_p}) \tag{69.1a}$$

and

$$I(g_{\alpha\beta};\ g_{\alpha\beta,\gamma_1\gamma_2};\ \ldots;\ g_{\alpha\beta,\gamma_1\ldots\gamma_p}) \tag{69.1b}$$

be respectively absolute affine and metric scalar differential invariants of order p, is that the systems

$$(a)\quad X^\mu_\nu(p)G = 0 \quad\text{and}\quad (b)\quad Y^\mu_\nu(p)I = 0 \tag{69.2}$$

be satisfied.

It follows from §66 that the above systems (69.2 (a)) and (69.2 (b)) are complete; hence the problem of determining all absolute affine and metric scalar differential invariants of any order p is identical with the problem of integrating a complete system of linear homogeneous partial differential equations(3).

70. Absolute metric differential invariants of order zero

Consider the expanded form of the equations (69.2 (b)) for $p = 0$, namely

$$(g_{\nu\beta}\delta^\mu_\alpha + g_{\alpha\nu}\delta^\mu_\beta)\frac{\partial I}{\partial g_{\alpha\beta}} = 0.$$

Multiplying these equations by $g^{\nu\sigma}$ and summing on the index ν, we obtain

$$\frac{\partial I}{\partial g_{\mu\sigma}}+\frac{\partial I}{\partial g_{\sigma\mu}}=0. \tag{70.1}$$

If we consider I to be expressed in terms of the $n(n+1)/2$ independent components $g_{\alpha\beta}(\alpha \leqq \beta)$, i.e. those components $g_{\alpha\beta}$ which are independent from the standpoint of the identities (40.1), then the equations (70.1) reduce to

$$\frac{\partial I}{\partial g_{\mu\sigma}}=0 \quad (\mu \leqq \sigma).$$

Hence there exists no absolute metric scalar differential invariant of order zero.

71. General theorems on the independence of the differential equations

We shall now consider the problem of determining all functionally independent invariants (69.1a) and (69.1b) when these invariants are taken as functions of components A and g which are independent from the standpoint of the complete sets of identities (41.1) and (41.2), and (41.10) and (41.11), respectively; in what we shall now have to say the word *independent* will be applied to the components A and g in this sense rather than in the sense of § 67. Also it will be assumed that the value of p in (69.2 (b)) is $\geqq 2$, since the case $p=0$ has been disposed of in § 70 and the case $p=1$ does not arise.

It is seen that the quantities (68.3), (68.7) and (68.12) satisfy identities of the form

$$\begin{pmatrix}\alpha\mu\\ \beta\gamma\delta\nu\end{pmatrix}=\begin{pmatrix}\alpha\mu\\ \gamma\beta\delta\nu\end{pmatrix},\quad \begin{pmatrix}\alpha\mu\\ \beta\gamma\delta\nu\end{pmatrix}+\begin{pmatrix}\alpha\mu\\ \gamma\delta\beta\nu\end{pmatrix}+\begin{pmatrix}\alpha\mu\\ \delta\beta\gamma\nu\end{pmatrix}=0, \tag{71.1a}$$

and

$$\left\{\begin{aligned}&\begin{bmatrix}\mu\\ \alpha\beta\nu\end{bmatrix}=\begin{bmatrix}\mu\\ \beta\alpha\nu\end{bmatrix},\quad \begin{bmatrix}\mu\\ \alpha\beta\gamma\delta\nu\end{bmatrix}=\begin{bmatrix}\mu\\ \beta\alpha\gamma\delta\nu\end{bmatrix}=\begin{bmatrix}\mu\\ \alpha\beta\delta\gamma\nu\end{bmatrix},\\ &\begin{bmatrix}\mu\\ \alpha\beta\gamma\delta\nu\end{bmatrix}+\begin{bmatrix}\mu\\ \alpha\gamma\delta\beta\nu\end{bmatrix}+\begin{bmatrix}\mu\\ \alpha\delta\beta\gamma\nu\end{bmatrix}=0,\end{aligned}\right. \tag{71.1b}$$

these identities evidently corresponding to the identities (41.7), (40.1), (41.14) and (41.15), respectively. Analogous identities are of course satisfied by the quantities (68.14) and (68.16).

The knowledge of the number of independent equations in the two systems (69.2) is necessary for the determination of the numbers of functionally independent absolute affine and metric differential invariants of order p. We proceed therefore to demonstrate the following general theorems.

Theorem I. *If the differential equations* (69·2 (a)) *or* (69·2 (b)) *for the determination of the absolute affine or metric scalar differential invariants of order p are independent for any affine or metric space, those for the determination of the invariants of order* $p+1$ *are also independent.*

To prove this theorem we observe that by hypothesis the equations (69.2 (a)) are independent. But the system

$$X^\mu_\nu(p+1)\,G=0$$

has the same number of equations as (69.2 (a)) and can in fact be formed from (69.2 (a)) merely by adding

$$\begin{pmatrix} \alpha\mu \\ \beta\gamma\delta_1\ldots\delta_{p+1}\nu \end{pmatrix} \frac{\partial G}{\partial A^\alpha_{\beta\gamma\delta_1\ldots\delta_{p+1}}}$$

to the left members of the corresponding equations (69.2 (a)). As a similar remark applies to the system (69.2 (b)), the above theorem therefore follows immediately.

THEOREM II A. *If the differential equations* (69·2 (a)) *for the determination of the absolute affine scalar differential invariants of order* 1 *are dependent for every affine space of n dimensions, they are dependent for every affine space of $n-1$ dimensions.*

By the hypothesis of this theorem the equations

$$\begin{pmatrix} \alpha\mu \\ \beta\gamma\delta\nu \end{pmatrix} \frac{\partial G}{\partial A^\alpha_{\beta\gamma\delta}}=0 \tag{71.2}$$

are dependent. Hence the linear homogeneous equations

$$\begin{pmatrix} \alpha\mu \\ \beta\gamma\delta\nu \end{pmatrix} \eta^\nu_\mu=0 \tag{71.3}$$

admit solutions η^ν_μ not all identically zero. Owing to the remark about the independent variables A made at the beginning of this section, the indices $\alpha, \beta, \gamma, \delta$ in (71.3) are to be considered to assume values $1, \ldots, n$ corresponding to those of $\alpha, \beta, \gamma, \delta$ in the set of variables $A^\alpha_{\beta\gamma\delta}$ which are taken as independent. However, the coefficients of η^ν_μ in (71.3) satisfy the identities (71.1a) which correspond exactly to the identities (41.7) satisfied by the components $A^\alpha_{\beta\gamma\delta}$. It follows therefore that the equations (71.3) hold for *all* values of the indices $\alpha, \beta, \gamma, \delta$.

We can write (71.3) explicitly in the form

$$A^\mu_{\beta\gamma\delta}\eta^\alpha_\mu - A^\alpha_{\nu\gamma\delta}\eta^\nu_\beta - A^\alpha_{\beta\nu\delta}\eta^\nu_\gamma - A^\alpha_{\beta\gamma\nu}\eta^\nu_\delta=0. \tag{71.4}$$

We now choose a particular affinely connected space of n dimensions in a manner determined by the following scheme*

$$\begin{aligned}
&\Gamma^n_{\beta\gamma}=0 && (\beta,\gamma=1,\ \ldots,\ n),\\
&\Gamma^\alpha_{n\gamma}=0 && (\alpha,\gamma=1,\ \ldots,\ n-1),\\
&\Gamma^\alpha_{nn}=\tfrac{3}{2}x^\alpha && (\alpha=1,\ \ldots,\ n-1),\\
&\Gamma^\alpha_{\beta\gamma}=\psi^\alpha_{\beta\gamma}(x^1,\ \ldots,\ x^{n-1}) && (\alpha,\beta,\gamma=1,\ \ldots,\ n-1),
\end{aligned} \tag{71.5}$$

where any $\psi^\alpha_{\beta\gamma}$ for $\alpha, \beta, \gamma=1, \ldots, n-1$ is an arbitrary analytic function of the

* It is to be understood that indices assume the range $1, \ldots, n$ unless a different range is specified.

variables $x^1, \ldots, x^{n-1}$. In the proof of our theorem we shall need the result of the following lemma.

LEMMA. *The particular components*

$$A^{\alpha}_{\beta\gamma\delta} \quad (\alpha, \beta, \gamma, \delta = 1, 2, \ldots, n-1) \tag{71.6}$$

of the normal tensor A of the special n-dimensional affinely connected space determined by the scheme (71.5) *constitute the totality of the components for the normal tensor A of the general affinely connected space of $n-1$ dimensions.*

The expression for the $A^{\alpha}_{\beta\gamma\delta}$ in terms of the $\Gamma^{\alpha}_{\beta\gamma}$ and their derivatives is given by the formula (35.7), i.e.

$$A^{\alpha}_{\beta\gamma\delta} = \frac{\partial \Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}} - \tfrac{1}{3}P\left(\frac{\partial \Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}} - 2\Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta}\right) - \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta} - \Gamma^{\alpha}_{\beta\mu}\Gamma^{\mu}_{\gamma\delta}, \tag{71.7}$$

where the symbol P denotes the sum of the terms obtainable from the ones inside the parenthesis by permuting the set of *subscripts* β, γ, δ cyclically. By means of (71.5) and (71.7) we get

$$A^{\alpha}_{\beta\gamma\delta} = \frac{\partial \Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}} - \tfrac{1}{3}P\left(\frac{\partial \Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}} - 2\sum_{\mu=1}^{n-1}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\mu}_{\gamma\delta}\right) - \sum_{\mu=1}^{n-1}(\Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta} + \Gamma^{\alpha}_{\beta\mu}\Gamma^{\mu}_{\gamma\delta}), \tag{71.8}$$

where $\alpha, \beta, \gamma, \delta = 1, 2, \ldots, n-1$. The lemma now follows immediately on observing that the components of the $\Gamma^{\alpha}_{\beta\gamma}$ in (71.8) depend only on $x^1, x^2, \ldots, x^{n-1}$ as is seen by a reference to the last set of equations in (71.5).

From (71.7) we have

$$A^{n}_{\beta\gamma\delta} = \frac{\partial \Gamma^{n}_{\beta\gamma}}{\partial x^{\delta}} - \tfrac{1}{3}P\left(\frac{\partial \Gamma^{n}_{\beta\gamma}}{\partial x^{\delta}} - 2\Gamma^{n}_{\mu\beta}\Gamma^{\mu}_{\gamma\delta}\right) - \Gamma^{n}_{\beta\mu}\Gamma^{\mu}_{\gamma\delta} - \Gamma^{n}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta}.$$

But by (71.5), $\Gamma^{n}_{\beta\gamma} = 0$ and hence

$$A^{n}_{\beta\gamma\delta} = 0. \tag{71.9}$$

By (71.7) we have also

$$A^{\alpha}_{n\gamma\delta} = \frac{\partial \Gamma^{\alpha}_{n\gamma}}{\partial x^{\delta}} - \tfrac{1}{3}P\left(\frac{\partial \Gamma^{\alpha}_{n\gamma}}{\partial x^{\delta}} - 2\Gamma^{\alpha}_{\mu n}\Gamma^{\mu}_{\gamma\delta}\right) - \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{n\delta} - \Gamma^{\alpha}_{n\mu}\Gamma^{\mu}_{\gamma\delta}. \tag{71.10}$$

Hence by (71.5) the relations (71.10) yield

$$A^{\alpha}_{n\gamma\delta} = 0 \quad (\alpha, \gamma, \delta = 1, 2, \ldots, n-1). \tag{71.11}$$

By employing the complete set of identities (41.7) and relations (71.11) we get

$$A^{\alpha}_{\beta n\delta} = 0, \quad A^{\alpha}_{\beta\gamma n} = 0 \quad (\alpha, \beta, \gamma, \delta = 1, 2, \ldots, n-1). \tag{71.12}$$

From relations (71.9), (71.11), and (71.12) it follows that equations (71.4) reduce to the equations

$$\sum_{\mu=1}^{n-1}(A^{\mu}_{\beta\gamma\delta}\eta^{\alpha}_{\mu} - A^{\alpha}_{\mu\gamma\delta}\eta^{\mu}_{\beta} - A^{\alpha}_{\beta\mu\delta}\eta^{\mu}_{\gamma} - A^{\alpha}_{\beta\gamma\mu}\eta^{\mu}_{\delta}) = 0 \tag{71.13}$$

$$(\alpha, \beta, \gamma, \delta = 1, 2, \ldots, n-1).$$

It is clear that, if η^ν_μ for $\mu, \nu = 1, 2, \ldots, n-1$ are not all identically zero, the differential equations for the determination of the differential invariants for every $(n-1)$-dimensional affinely connected space are dependent. We shall give now a proof by *reductio ad absurdum* that

$$\eta^\nu_\mu \not\equiv 0 \quad \text{for} \quad \mu, \nu = 1, 2, \ldots, n-1.$$

Suppose then that

(71.14) $$\eta^\nu_\mu = 0 \quad (\mu, \nu = 1, 2, \ldots, n-1).$$

From equations (71.4) for $\beta, \gamma = n$; $\alpha, \delta = 1, 2, \ldots, n-1$ we obtain

(71.15) $$A^\mu_{nn\delta}\eta^\alpha_\mu - A^\alpha_{\mu n\delta}\eta^\mu_n - A^\alpha_{n\mu\delta}\eta^\mu_n - A^\alpha_{nn\mu}\eta^\mu_\delta = 0 \quad (\alpha, \delta = 1, 2, \ldots, n-1).$$

By (41.7), (71.9), (71.11) and (71.14) the relations (71.15) yield

(71.16) $$A^\alpha_{nn\delta}\eta^n_n = 0 \quad (\alpha, \delta = 1, 2, \ldots, n-1).$$

On employing relations (71.5) and (71.7) we get by calculation

(71.17) $$A^\alpha_{nn\delta} = \delta^\alpha_\delta + \sum_{\mu=1}^{n-1} x^\mu \Gamma^\alpha_{\mu\delta} \quad (\alpha, \delta = 1, 2, \ldots, n-1).$$

It is clear from (71.5) that $A^\alpha_{nn\delta}$ for $\alpha, \delta = 1, 2, \ldots, n-1$ depends only on the points of the $(n-1)$-dimensional affinely connected space. If $x^\alpha = 0$ for $\alpha = 1, 2, \ldots, n-1$, then the determinant of $n-1$ rows

(71.18) $$|A^\alpha_{nn\delta}| = 1.$$

Hence in the neighbourhood of the values $x^\alpha = 0$ for $\alpha = 1, 2, \ldots, n-1$ we have

(71.19) $$|A^\alpha_{nn\delta}| \neq 0 \quad (\alpha, \delta = 1, 2, \ldots, n-1).$$

We have also

(71.20) $$A^\alpha_{nn\delta} \neq 0 \quad \text{when} \quad \alpha = \delta \quad (\alpha, \delta = 1, 2, \ldots, n-1).$$

From (71.16) and (71.20) we get

(71.21) $$\eta^n_n = 0.$$

Putting $\alpha, \beta, \gamma = n$ in (71.4) and employing conditions (71.9) we see that

(71.22) $$\sum_{\mu=1}^{n-1} A^\mu_{nn\delta}\eta^n_\mu = 0 \quad (\delta = 1, 2, \ldots, n-1).$$

By (71.19) it follows that

(71.23) $$\eta^n_\alpha = 0 \quad (\alpha = 1, 2, \ldots, n-1).$$

Now consider equations (71.4) for $\alpha, \beta, \gamma = 1, 2, \ldots, n-1$ and $\delta = n$. Then by means of (71.9), (71.11), (71.12), (71.14), (71.21) and (71.23) we have

(71.24) $$\sum_{\nu=1}^{n-1} A^\alpha_{\beta\gamma\nu}\eta^\nu_n = 0 \quad (\alpha, \beta, \gamma = 1, 2, \ldots, n-1).$$

In general there exists an $(n-1)$-rowed determinant*

(71.25) $$D \neq 0,$$

* The existence of such a determinant D may easily be inferred from the fact that the $\Gamma^\alpha_{\beta\gamma}$ with indices $1, 2, \ldots, n-1$ may be chosen so that the A's at any point have arbitrary values subject only to the complete set of identities for the A (see § 46).

which is formed from the coefficients A in (71.24), and hence in general

(71.26) $$\eta_n^\alpha = 0 \quad (\alpha = 1, 2, \ldots, n-1).$$

Hence contrary to our hypothesis about (71.2) we have

(71.27) $$\eta_\alpha^\sigma = 0$$

under condition (71.25). Hence under the condition (71.25) equations (71.13) have solutions $\eta \neq 0$. This means that under the condition (71.25) there exists a number R of determinants of the matrix of (71.13) which vanish in the variables A. Now these R determinants and the determinant D are polynomials in the variables A. Furthermore these R determinants vanish for all values of the variables A for which D does not vanish and hence by a theorem in algebra* the R determinants vanish identically irrespective of whether D vanishes or not. Hence (71.13) in all cases possesses solutions $\eta \neq 0$ and thus our theorem has been proved.

We now pass to the consideration of the analogous theorem for the case of the metric space.

THEOREM IIB. *If the differential equations* (69·2 (*b*)) *for the determination of the absolute metric scalar differential invariants of order two are dependent for every metric space of n dimensions, they are dependent for every metric space of* $n-1$ *dimensions.*

By the hypothesis of this theorem the equations

(71.28) $$\begin{bmatrix} \mu \\ \alpha\beta\nu \end{bmatrix} \frac{\partial I}{\partial g_{\alpha\beta}} + \begin{bmatrix} \mu \\ \alpha\beta\gamma\delta\nu \end{bmatrix} \frac{\partial I}{\partial g_{\alpha\beta,\gamma\delta}} = 0$$

are dependent. Consequently the linear homogeneous equations

(71.29) $$(a) \begin{bmatrix} \mu \\ \alpha\beta\nu \end{bmatrix} \eta_\mu^\nu = 0, \quad (b) \begin{bmatrix} \mu \\ \alpha\beta\gamma\delta\nu \end{bmatrix} \eta_\mu^\nu = 0$$

admit solutions η which are not all identically zero, where the indices α, β in (71.29 (*a*)) and α, β, γ, δ in (71.29 (*b*)) correspond to the indices of sets of independent components $g_{\alpha\beta}$ and $g_{\alpha\beta,\gamma\delta}$, respectively. It follows, however, that (71.29) holds for all values of the indices α, β, γ, δ on account of the existence of the identities (71.1*b*) corresponding to the result obtained in the affine case.

The expanded form of (71.29) is

(71.30) $$g_{\sigma\beta}\eta_\alpha^\sigma + g_{\alpha\sigma}\eta_\beta^\sigma = 0,$$

and

(71.31) $$g_{\sigma\beta,\gamma\delta}\eta_\alpha^\sigma + g_{\alpha\sigma,\gamma\delta}\eta_\beta^\sigma + g_{\alpha\beta,\sigma\delta}\eta_\gamma^\sigma + g_{\alpha\beta,\gamma\sigma}\eta_\delta^\sigma = 0.$$

Let us now choose

(71.32) $$g_{\alpha\beta} = \psi_{\alpha\beta}(x^1, \ldots, x^{n-1}) \quad (\alpha, \beta = 1, 2, \ldots, n-1),$$

and

(71.33) $$g_{n\alpha} = \begin{cases} 0 \text{ if } \alpha \neq n, \\ 1 \text{ if } \alpha = n, \end{cases}$$

* Cf. Bocher, *Introduction to Higher Algebra*, p. 8.

where $\psi_{\alpha\beta}$ denotes an arbitrary analytic function of its arguments $x^1, \ldots, x^{n-1}$ such that the determinant $|\psi_{\alpha\beta}| \neq 0$. With this selection of the components $g_{\alpha\beta}$ it follows that the components $g_{\alpha\beta,\gamma\delta}$, where $\alpha, \beta, \gamma, \delta = 1, 2, \ldots, n-1$, are derivable as the second extension of a tensor whose components are given by (71.32). It also follows that

$$(71.34)\qquad \begin{aligned} &g_{n\alpha,\beta\gamma} = g_{\alpha\beta,n\gamma} = 0 \quad (\alpha, \beta, \gamma = 1, 2, \ldots, n-1),\\ &g_{nn,\beta\gamma} = g_{n\beta,n\gamma} = g_{\beta\gamma,nn} = 0 \quad (\beta, \gamma = 1, 2, \ldots, n-1), \end{aligned}$$

when use is made of the formula (35.9). From (71.30) and (71.33) we obtain

$$(71.35)\qquad \sum_{\sigma=1}^{n-1} (g_{\sigma\beta}\eta^\sigma_\alpha + g_{\alpha\sigma}\eta^\sigma_\beta) = 0 \quad (\alpha, \beta = 1, 2, \ldots, n-1),$$

and

$$(71.36)\qquad \sum_{\sigma=1}^{n-1} g_{\sigma\beta}\eta^\sigma_n + \eta^n_\beta = 0 \quad (\beta = 1, 2, \ldots, n-1),$$

where the latter equations are obtained by putting $\alpha = n$ in (71.30). Similarly from (71.31) and (71.34) we find that

$$(71.37)\qquad \sum_{\sigma=1}^{n-1} (g_{\sigma\beta,\gamma\delta}\eta^\sigma_\alpha + g_{\alpha\sigma,\gamma\delta}\eta^\sigma_\beta + g_{\alpha\beta,\sigma\delta}\eta^\sigma_\gamma + g_{\alpha\beta,\gamma\sigma}\eta^\sigma_\delta) = 0$$
$$(\alpha, \beta, \gamma, \delta = 1, 2, \ldots, n-1),$$

and

$$(71.38)\qquad \sum_{\sigma=1}^{n-1} g_{\sigma\beta,\gamma\delta}\eta^\sigma_n = 0 \quad (\beta, \gamma, \delta = 1, 2, \ldots, n-1).$$

Finally, by putting $\alpha = \beta = n$ in (71.30), we have

$$(71.39)\qquad \eta^n_n = 0 \quad (\text{not summed for } n).$$

From what has been said we see that the equations (71.35) and (71.37) correspond to the equations (71.30) and (71.31) for the general metric space of $n-1$ dimensions. If the differential equations (71.28) are independent for a metric space of $n-1$ dimensions, then the only solution of the equations (71.35) and (71.37) is

$$(71.40)\qquad \eta^\sigma_\alpha = 0 \quad (\sigma, \alpha = 1, 2, \ldots, n-1).$$

It is evident as for the affine case that in general there exists a determinant D of order $n-1$ formed from the coefficients of η^σ_n in (71.38) which does not vanish identically. In general then the quantities η^σ_n are zero, from which it follows by (71.36) that the quantities η^n_σ are also zero. Under the assumption (71.40) and the assumption of the existence of a determinant D it follows that all quantities $\eta^\nu_\mu = 0$, which is contrary to the hypothesis that the equations (71.28) are dependent. Finally the apparent restriction in the assumption of the existence of the determinant D can be removed by the same argument used in the affine case. This completes the proof of Theorem IIB.

72. Number of independent differential equations. Affine case

We now turn to the demonstration of some theorems for the general two and three dimensional spaces with symmetric affine connections. By means of these theorems and those proved in the preceding section we shall be enabled to obtain definite information regarding the independence of the differential equations for the determination of the absolute scalar differential invariants of a general n-dimensional space of symmetric affine connection.

The differential equations for the determination of the differential invariants of order 1 for the general symmetric affinely connected space of two dimensions turn out to be dependent. We can demonstrate, however, the following theorem for the general symmetric affinely connected space of two dimensions.

THEOREM IIIA. *The differential equations* (69·2 (a)) *for the determination of the absolute scalar differential invariants of order* $p \geqq 2$ *are independent for the general symmetric affinely connected space of two dimensions.*

To prove this theorem completely it is sufficient to show that the equations for the invariants of order $p = 2$ are independent, as is evident by a reference to Theorem I.

The differential equations are*

$$\left(\begin{matrix}\alpha\mu\\ \beta\gamma\delta\nu\end{matrix}\right)\frac{\partial G}{\partial A^{\alpha}_{\beta\gamma\delta}}+\left(\begin{matrix}\alpha\mu\\ \beta\gamma\delta\epsilon\nu\end{matrix}\right)\frac{\partial G}{\partial A^{\alpha}_{\beta\gamma\delta\epsilon}}=0. \tag{72.1}$$

We see from the formula for $A(n,p)$ in § 54 that the number of algebraically independent components of the normal tensors $A^{\alpha}_{\beta\gamma\delta}$ and $A^{\alpha}_{\beta\gamma\delta\epsilon}$ for the two dimensional symmetric affinely connected space is given by

$$A(2,1)=4 \quad\text{and}\quad A(2,2)=8,$$

respectively. These twelve components can be chosen as

$$\begin{cases} A^{1}_{211}=\alpha,\ A^{2}_{122}=\beta,\ A^{1}_{122}=\gamma,\ A^{2}_{211}=\delta,\\ A^{1}_{1122}=a,\ A^{2}_{1122}=b,\ A^{1}_{1212}=c,\ A^{2}_{1212}=d,\\ A^{1}_{1112}=e,\ A^{2}_{1112}=f,\ A^{1}_{2212}=g,\ A^{2}_{2212}=h,\end{cases} \tag{72.2}$$

and by calculation from the identities (41.7) and (41.9) we get

$$\begin{cases} A^{1}_{112}=-2\alpha,\ A^{2}_{221}=-2\beta,\ A^{1}_{221}=-2\gamma,\ A^{2}_{112}=-2\delta,\\ A^{1}_{2211}=-(a+4c),\ A^{1}_{1211}=-e,\ A^{1}_{1222}=-g,\\ A^{2}_{2211}=-(b+4d),\ A^{2}_{1211}=-f,\ A^{2}_{1222}=-h.\end{cases} \tag{72.3}$$

Let us now express the invariant G in (72.1) as well as the quantities

$$\left(\begin{matrix}\alpha\mu\\ \beta\gamma\delta\nu\end{matrix}\right) \quad\text{and}\quad \left(\begin{matrix}\alpha\mu\\ \beta\gamma\delta\epsilon\nu\end{matrix}\right)$$

* In the proof of this theorem we shall understand that subscripts and superscripts take on only the values 1, 2.

in terms of the independent components $A^{\alpha}_{\beta\gamma\delta}$ and $A^{\alpha}_{\beta\gamma\delta\epsilon}$ in (72.2) alone. Then all the derivatives of G with respect to the dependent components $A^{\alpha}_{\beta\gamma\delta}$ and $A^{\alpha}_{\beta\gamma\delta\epsilon}$ vanish, and after some calculation the equations (72.1) assume a form which can be represented by the matrix of the coefficients given in Table I. The determinant formed from the first, second, seventh and eighth rows of this matrix does not vanish identically and hence our theorem is proved.

TABLE I

$(\mu, \nu) =$	(1, 1)	(2, 1)	(1, 2)	(2, 2)
$\partial G/\partial \alpha$	α	$-\delta$	$-\gamma$	α
$\partial G/\partial \beta$	β	$-\delta$	$-\gamma$	β
$\partial G/\partial \gamma$	0	$-(\alpha+\beta)$	0	2γ
$\partial G/\partial \delta$	2δ	0	$-(\alpha+\beta)$	0
$\partial G/\partial a$	a	$(2e-b)$	$-2g$	$2a$
$\partial G/\partial b$	$2b$	$2f$	$-(a+2h)$	b
$\partial G/\partial c$	c	$-d$	0	$2c$
$\partial G/\partial d$	$2d$	0	$-c$	d
$\partial G/\partial e$	$2e$	$-f$	$(a+2c)$	e
$\partial G/\partial f$	$3f$	0	$(b+2d-e)$	0
$\partial G/\partial g$	0	$-(a+2c+h)$	0	$3g$
$\partial G/\partial h$	h	$-(b+2d)$	$-g$	$2h$

THEOREM IV A. *The differential equations (71·2) for the determination of the absolute scalar differential invariants of order one for the general symmetric affinely connected space of three dimensions are independent.**

As before we find that the number of algebraically independent components of the normal tensor $A^{\alpha}_{\beta\gamma\delta}$ is given by

$$A\,(3, 1) = 24. \tag{72.4}$$

By means of the complete set of identities (41.7) it is easily seen that the following twenty-four components $A^{\alpha}_{\beta\gamma\delta}$ are algebraically independent:

$$\begin{cases} A^{\alpha}_{121} = a^{\alpha}, \ A^{\alpha}_{131} = b^{\alpha}, \ A^{\alpha}_{212} = c^{\alpha}, \ A^{\alpha}_{232} = d^{\alpha}, \\ A^{\alpha}_{313} = e^{\alpha}, \ A^{\alpha}_{323} = f^{\alpha}, \ A^{\alpha}_{231} = g^{\alpha}, \ A^{\alpha}_{312} = h^{\alpha}. \end{cases} \tag{72.5}$$

The non-vanishing dependent components not equal to those in (72.5) by the first set of identities (41.7) are

$$\begin{array}{l} -2a^{\alpha}, \ A^{\alpha}_{113} = -2b^{\alpha}, \ A^{\alpha}_{221} = -2c^{\alpha}, \ A^{\alpha}_{223} = -2d^{\alpha}, \\ A^{\alpha}_{331} = -2e^{\alpha}, \ A^{\alpha}_{332} = -2f^{\alpha}, \ A^{\alpha}_{123} = A^{\alpha}_{213} = -(g^{\alpha}+h^{\alpha}). \end{array} \tag{72.6}$$

By expressing the equations (71.2) in terms of the independent components (72.5) in the same way as was done for the case $n=2$, we could construct a table analogous to Table I, which however we will not write out in full. To show that not all ninth order determinants of this matrix vanish identically, we will show that one of these determinants does not vanish for a particular set of values for the $A^{\alpha}_{\beta\gamma\delta}$.

* In the proof of this theorem we shall understand that all subscripts and superscripts can take on values 1, 2, 3.

Putting $\qquad a^2 = d^3 = e^1 = 1, \quad \text{other components} = 0$

in the matrix, and selecting the rows of the matrix corresponding to the coefficients of

$$\frac{\partial G}{\partial a^1}, \ \frac{\partial G}{\partial a^2}, \ \frac{\partial G}{\partial a^3}, \ \frac{\partial G}{\partial b^2}, \ \frac{\partial G}{\partial c^3}, \ \frac{\partial G}{\partial d^3}, \ \frac{\partial G}{\partial e^1}, \ \frac{\partial G}{\partial g^2}, \ \frac{\partial G}{\partial g^3},$$

we get the determinant

$$(72.7) \qquad \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Thus the rank of the matrix for the general symmetric affinely connected space of three dimensions is 9 and hence Theorem IVA is proved.

The following theorem is a consequence of Theorems I, II A, and IVA.

THEOREM V A. *The differential equations* (69·2 (a)) *for the determination of the absolute scalar differential invariants of order* $p \geqq 1$ *for the general symmetric affinely connected space of* $n \geqq 3$ *dimensions are independent.*

73. NUMBER OF INDEPENDENT DIFFERENTIAL EQUATIONS. METRIC CASE

We will first consider the equations (69.2 (b)) for $n=2$ and $p=3$, i.e. the equations

$$(73.1) \qquad \begin{bmatrix} \mu \\ \alpha\beta\nu \end{bmatrix} \frac{\partial I}{\partial g_{\alpha\beta}} + \begin{bmatrix} \mu \\ \alpha\beta\gamma\delta\nu \end{bmatrix} \frac{\partial I}{\partial g_{\alpha\beta,\gamma\delta}} + \begin{bmatrix} \mu \\ \alpha\beta\gamma\delta\epsilon\nu \end{bmatrix} \frac{\partial I}{\partial g_{\alpha\beta,\gamma\delta\epsilon}} = 0,$$

where the indices take on the values 1, 2. For this case the independent components will be taken as

$$(73.2) \qquad \begin{cases} g_{11} = \alpha, \ g_{12} = \beta, \ g_{22} = \gamma, \\ g_{12,12} = \delta, \ g_{12,122} = \epsilon, \ g_{12,112} = \zeta. \end{cases}$$

Then the dependent components which are not zero and which are not equal among themselves on account of (40.1), (41.10), (41.11) and (41.16) are

$$g_{21} = \beta, \ g_{22,112} = -\epsilon, \ g_{22,111} = -3\zeta,$$
$$g_{11,22} = -2\delta, \ g_{11,122} = -\zeta, \ g_{11,222} = -3\epsilon.$$

By expressing the invariant I and the coefficients in equations (73.1) in terms of the independent components (73.2), in the same way as was done

for the affine case, we obtain the matrix of coefficients given in Table II. The rank of this matrix is four in general, since, for example, the determinant

TABLE II

$\binom{\mu}{\nu}$	$\frac{\partial I}{\partial \alpha}$	$\frac{\partial I}{\partial \beta}$	$\frac{\partial I}{\partial \gamma}$	$\frac{\partial I}{\partial \delta}$	$\frac{\partial I}{\partial \epsilon}$	$\frac{\partial I}{\partial \zeta}$
$\binom{1}{1}$	2α	β	0	2δ	2ϵ	3ζ
$\binom{1}{2}$	2β	γ	0	0	0	ϵ
$\binom{2}{1}$	0	α	2β	0	ζ	0
$\binom{2}{2}$	0	β	2γ	2δ	3ϵ	2ζ

formed from columns 1, 2, 3, 5 does not vanish identically. Thus we arrive at the following

THEOREM IIIB. *The differential equations* (73·1) *for the determination of the absolute scalar differential invariants of order* 3 *for the general metric space of two dimensions are independent.*

TABLE III

$\binom{\mu}{\nu}$	$\frac{\partial I}{\partial \alpha}$	$\frac{\partial I}{\partial \beta}$	$\frac{\partial I}{\partial \gamma}$	$\frac{\partial I}{\partial \delta}$	$\frac{\partial I}{\partial \epsilon}$	$\frac{\partial I}{\partial \zeta}$	$\frac{\partial I}{\partial a}$	$\frac{\partial I}{\partial b}$	$\frac{\partial I}{\partial c}$	$\frac{\partial I}{\partial d}$	$\frac{\partial I}{\partial e}$	$\frac{\partial I}{\partial f}$
$\binom{1}{1}$	2α	β	γ	0	0	0	$2a$	$2b$	$2c$	d	e	0
$\binom{1}{2}$	2β	δ	ϵ	0	0	0	0	$-d$	$2e$	0	f	0
$\binom{1}{3}$	2γ	ϵ	ζ	0	0	0	$2d$	$-e$	0	f	0	0
$\binom{2}{1}$	0	α	0	2β	γ	0	0	0	0	$-b$	c	$2e$
$\binom{2}{2}$	0	β	0	2δ	ϵ	0	$2a$	b	0	$2d$	e	$2f$
$\binom{2}{3}$	0	γ	0	2ϵ	ζ	0	$2b$	c	0	$-e$	0	0
$\binom{3}{1}$	0	0	α	0	β	2γ	0	0	0	a	$-b$	$2d$
$\binom{3}{2}$	0	0	β	0	δ	2ϵ	0	a	$2b$	0	$-d$	0
$\binom{3}{3}$	0	0	γ	0	ϵ	2ζ	0	b	$2c$	d	$2e$	$2f$

We now consider the equations (71.28) for $n=3$. In this case the independent components $g_{\alpha\beta}$ and $g_{\alpha\beta,\gamma\delta}$ can be selected as

$$\begin{cases} g_{11}=\alpha, \; g_{12}=\beta, \; g_{13}=\gamma, \; g_{22}=\delta, \; g_{23}=\epsilon, \; g_{33}=\zeta, \\ g_{12,12}=a, \; g_{12,13}=b, \; g_{13,13}=c, \\ g_{12,23}=d, \; g_{13,23}=e, \; g_{23,23}=f. \end{cases} \tag{73.3}$$

The remaining components which do not vanish identically and which are not identical among themselves on account of (40.1), (41.1) and (41.15) are then given by

$$
(73.4) \qquad \begin{cases} g_{21}=\beta, \ g_{31}=\gamma, \ g_{32}=\epsilon, \\ g_{11,22}=-2a, \ g_{11,23}=-2b, \ g_{11,33}=-2c, \\ g_{13,22}=-2d, \ g_{12,33}=-2e, \ g_{22,33}=-2f. \end{cases}
$$

By a method entirely similar to the one employed in treating equations (73.1) we obtain the matrix of the coefficients of (71.28) which is given in Table III. The rank of the matrix in Table III is nine in general as may be seen most easily by assigning special numerical values to the components α, β, ..., f and then selecting from the matrix a determinant of order nine which does not vanish. Thus if we put $\beta=c=d=1$ and take the remaining components in the set α, ..., f as zero, the determinant

$$
(73.5) \qquad \begin{matrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 & 0 & 2 & 1 \end{matrix}
$$

formed from columns 1, 2, 3, 4, 5, 7, 8, 9, 10 of the Table III has the value -32. This establishes the following theorem.

THEOREM IV B. *The differential equations* (71·28) *for the determination of the absolute scalar differential invariants of order* 2 *for the general metric space of three dimensions are independent.*

On the basis of Theorems I, II B, III B and IV B we have the following general result.

THEOREM V B. *For the general metric space of n dimensions there are n^2 independent differential equations* (69.2 (b)) *for the determination of the absolute scalar differential invariants of order p if $n \geqq 2$, $p \geqq 3$ or if $n \geqq 3$, $p \geqq 2$.*

74. Exceptional case of two dimensions

Let us now consider the equations (71.2) for $n=2$. If we choose the independent components $A^{\alpha}_{\beta\gamma\delta}$ as in § 72, the matrix of the coefficients is given by the first four rows of Table I. Hence equations (71.2) become

$$
(74.1)\qquad
\begin{aligned}
&\alpha\frac{\partial G}{\partial\alpha}+\beta\frac{\partial G}{\partial\beta} &&+2\delta\frac{\partial G}{\partial\delta}=0,\\
&\delta\frac{\partial G}{\partial\alpha}+\delta\frac{\partial G}{\partial\beta}+(\alpha+\beta)\frac{\partial G}{\partial\gamma} &&=0,\\
&\gamma\frac{\partial G}{\partial\alpha}+\gamma\frac{\partial G}{\partial\beta} &&+(\alpha+\beta)\frac{\partial G}{\partial\delta}=0,\\
&\alpha\frac{\partial G}{\partial\alpha}+\beta\frac{\partial G}{\partial\beta}\quad+2\gamma\frac{\partial G}{\partial\gamma} &&=0.
\end{aligned}
$$

The determinant of equations (74.1) is found to vanish identically so that the equations are not all independent; but *the first, second and fourth equations are independent.* Since the system (74.1) is complete, we know that there exists a solution G of these equations such that any function of G is a solution of (74.1) and any solution of (74.1) is a function of G (see § 66). Solving the equations (74.1) we find that this solution has the form

$$
(74.2)\qquad G(\alpha,\beta,\gamma,\delta)=\frac{(\alpha-\beta)^2}{\alpha\beta-\gamma\delta}.
$$

The invariant property of the above scalar $G(\alpha,\beta,\gamma,\delta)$ does not appear to be evident on inspecting its form. There is interest in verifying its invariant character directly from the transformation law of the components α, β, γ, δ in (74.2). We wish to show directly that

$$
(74.3)\qquad \frac{(\bar\alpha-\bar\beta)^2}{\bar\alpha\bar\beta-\bar\gamma\bar\delta}=\frac{(\alpha-\beta)^2}{\alpha\beta-\gamma\delta}.
$$

The equations relating the quantities $\bar\alpha$, $\bar\beta$, $\bar\gamma$, $\bar\delta$ with the components $A^{\alpha}_{\beta\gamma\delta}$ are

$$
(74.4)\qquad
\begin{cases}
\bar\alpha=A^{\nu}_{\alpha\beta\gamma}\,\bar u^{1}_{\nu}u^{\alpha}_{2}u^{\beta}_{1}u^{\gamma}_{1},\\
\bar\beta=A^{\nu}_{\alpha\beta\gamma}\,\bar u^{2}_{\nu}u^{\alpha}_{1}u^{\beta}_{2}u^{\gamma}_{2},\\
\bar\gamma=A^{\nu}_{\alpha\beta\gamma}\,\bar u^{1}_{\nu}u^{\alpha}_{1}u^{\beta}_{2}u^{\gamma}_{2},\\
\bar\delta=A^{\nu}_{\alpha\beta\gamma}\,\bar u^{2}_{\nu}u^{\alpha}_{2}u^{\beta}_{1}u^{\gamma}_{1}
\end{cases}
$$

(see § 67). Writing the components $A^{\nu}_{\alpha\beta\gamma}$ in the right-hand members of the first two equations (74.4) in terms of the quantities α, β, γ, δ, we obtain after some calculation

$$
\begin{aligned}
(74.5)\qquad (\bar\alpha-\bar\beta)=\Delta\,[\alpha(\bar u^{1}_{1}u^{1}_{1}+\bar u^{2}_{1}u^{1}_{2})-\beta(\bar u^{1}_{2}u^{2}_{1}+\bar u^{2}_{2}u^{2}_{2})&\\
+\delta(\bar u^{1}_{2}u^{1}_{1}+\bar u^{2}_{2}u^{1}_{2})-\gamma(\bar u^{1}_{1}u^{2}_{1}+\bar u^{2}_{1}u^{2}_{2})]&,
\end{aligned}
$$

where

$$
\Delta=\begin{vmatrix} u^{1}_{1} & u^{1}_{2}\\ u^{2}_{1} & u^{2}_{2}\end{vmatrix}.
$$

But

$$
(74.6)\qquad
\begin{cases}
\bar u^{1}_{1}u^{1}_{1}+\bar u^{2}_{1}u^{1}_{2}=1, & \bar u^{1}_{2}u^{2}_{1}+\bar u^{2}_{2}u^{2}_{2}=1,\\
\bar u^{1}_{2}u^{1}_{1}+\bar u^{2}_{2}u^{1}_{2}=0, & \bar u^{1}_{1}u^{2}_{1}+\bar u^{2}_{1}u^{2}_{2}=0.
\end{cases}
$$

Hence

$$
(74.7)\qquad \bar\alpha-\bar\beta=\Delta\,(\alpha-\beta).
$$

In a similar manner we find first that

$$
\bar\alpha\bar\beta-\bar\gamma\bar\delta=\Delta^{3}\,[(\alpha\bar u^{1}_{1}+\delta\bar u^{1}_{2})(\gamma\bar u^{2}_{1}+\beta\bar u^{2}_{2})-(\alpha\bar u^{2}_{1}+\delta\bar u^{2}_{2})(\gamma\bar u^{1}_{1}+\beta\bar u^{1}_{2})],
$$

which upon simplification becomes

$$
\bar\alpha\bar\beta-\bar\gamma\bar\delta=\Delta^{3}\,(\bar u^{1}_{1}\bar u^{2}_{2}-\bar u^{1}_{2}\bar u^{2}_{1})\,(\alpha\beta-\gamma\delta).
$$

Finally, noting that $\bar{u}_1^1\bar{u}_2^2-\bar{u}_2^1\bar{u}_1^2=1/\Delta$, we have

$$\bar{\alpha}\bar{\beta}-\bar{\gamma}\bar{\delta}=\Delta^2(\alpha\beta-\gamma\delta). \tag{74.8}$$

The relation (74.3) then follows from (74.7) and (74.8).

Incidentally we have shown that $(\alpha-\beta)$ and $(\alpha\beta-\gamma\delta)$ are relative scalar affine differential invariants of weights 1 and 2 respectively. Hence the absolute invariant $G(\alpha,\beta,\gamma,\delta)$ is formed by means of the ratio of two relative invariants each of which is of weight 2.

The differential equations (71.28) for the determination of the absolute scalar differential invariants of order two for a general metric space of two dimensions are dependent. In fact, let

$$g_{11}=\alpha,\ g_{12}=\beta,\ g_{22}=\gamma,\ g_{12,12}=\delta \tag{74.9}$$

be the four independent components in terms of which all the other $g_{\alpha\beta}$ and $g_{\alpha\beta,\gamma\delta}$ can be expressed. The remaining components $g_{\alpha\beta}$ and $g_{\alpha\beta,\gamma\delta}$, which are not identically zero, are given by

$$g_{21}=\beta,\ g_{21,12}=g_{21,21}=g_{12,21}=\delta,\ g_{11,22}=g_{22,11}=-2\delta.$$

If we consider the invariant I in (71.28) as well as the quantities

$$\begin{bmatrix}\mu\\ \alpha\beta\nu\end{bmatrix}\ \text{and}\ \begin{bmatrix}\mu\\ \alpha\beta\gamma\delta\nu\end{bmatrix}$$

to be expressed in terms of the independent components $g_{\alpha\beta}$ and $g_{\alpha\beta,\gamma\delta}$ in (74.9), then all the derivatives of I with respect to the dependent components $g_{\alpha\beta}$ and $g_{\alpha\beta,\gamma\delta}$ vanish, and the equations (71.28) for the metric space under consideration become

$$\begin{aligned}
2\alpha\frac{\partial I}{\partial\alpha}+\beta\frac{\partial I}{\partial\beta}\qquad\qquad\quad +2\delta\frac{\partial I}{\partial\delta}&=0,\\
2\beta\frac{\partial I}{\partial\alpha}+\gamma\frac{\partial I}{\partial\beta}\qquad\qquad\qquad\qquad\quad &=0,\\
\alpha\frac{\partial I}{\partial\beta}+2\beta\frac{\partial I}{\partial\gamma}\qquad\qquad &=0,\\
\beta\frac{\partial I}{\partial\beta}+2\gamma\frac{\partial I}{\partial\gamma}+2\delta\frac{\partial I}{\partial\delta}&=0.
\end{aligned} \tag{74.10}$$

The determinant of the equations (74.10) vanishes identically, which shows the dependence of these equations. However the determinant

$$\begin{vmatrix}2\alpha & \beta & 0\\ 2\beta & \gamma & 0\\ 0 & \alpha & 2\beta\end{vmatrix}=4\beta(\alpha\gamma-\beta^2)$$

does not vanish in general. Hence, *the first three equations in* (74.10) *are independent for a general metric space of two dimensions.*

It follows from the above result, since the system (74.10) is complete, that there exists a single analytic solution I of these equations and that any other analytic solution can be expressed as a suitable function of this solution I.

From formal considerations alone it is evident that this solution I of (74.10) can be taken as

$$I = g^{\alpha\beta} g^{\gamma\delta} g_{\alpha\beta,\gamma\delta} = 4\delta/(\beta^2 - \alpha\gamma),$$

with reference to the metric space of two dimensions.

75. Fundamental sets of absolute scalar differential invariants

We are now in a position to give the number of functionally independent affine or metric absolute scalar differential invariants of all orders p. In this connection we shall define two sets of differential invariants which will be called *the first and second fundamental sets of (affine or metric) differential invariants.*

Definition. *A first fundamental set of affine or metric scalar differential invariants of order p is a fundamental set of solutions* of the systems (69.2 (a)) or (69.2 (b)), each of which is of order p when considered as a differential invariant.*

The actual existence of a first fundamental set of invariants can easily be inferred. Let $\mathfrak{A}(n,p)$ denote the number of functionally independent solutions of the complete system (69.2 (a)). Then on account of Theorems IIIA and VA we have

$$\mathfrak{A}(n,p) = \sum_{\sigma=1}^{p} A(n,\sigma) - n^2 \quad (n \geqq 2,\ p \geqq 1;\ n \neq 2 \text{ if } p=1), \tag{75.1}$$

where $A(n,\sigma)$ is the number of algebraically independent components $A^{\alpha}_{\beta\gamma\delta_1\ldots\delta_\sigma}$ as given in § 54; in particular we have $\mathfrak{A}(2,1)=1$ from the result of § 74. By making use of a formula similar to (54.9) obtained from the identity satisfied by $K(n,p)$ we can write (75.1) in the form

$$\mathfrak{A}(n,p) = n\,[K(n,2)\,K(n+1,p) - K(n+1,p+2) + 1]. \tag{75.2}$$

This number $\mathfrak{A}(n,p)$ is precisely the number of absolute affine scalar differential invariants in a fundamental set of order p. In fact there must be at least one component of the normal tensor A with $p+2$ subscripts in at least one of the functionally independent solutions of the partial differential equations (69.2 (a)); for, if not, the equations (69.2 (a)) where p is replaced by $p-1$ would have the same number of functionally independent solutions as (69.2 (a)), which is impossible since

$$\mathfrak{A}(n,p) > \mathfrak{A}(n,p-1).$$

We can therefore pick $\mathfrak{A}(n,p)$ suitable independent functions, each of which actually involves at least one component $A^{\alpha}_{\beta\gamma\delta_1\ldots\delta_p}$, as solutions of (69.2 (a)).

* By a fundamental set of solutions of a system of differential equations we understand a set of functionally independent solutions such that any solution of the equations can be expressed as a function of the members of the set.

Similar remarks apply to the metric case; thus if we denote by $\mathfrak{G}(n,p)$ the number of functionally independent solutions of the complete system (69.2 (b)), it follows that the number $\mathfrak{G}(n,p)$ is likewise the number of invariants in a first fundamental set of absolute metric scalar differential invariants of order p. By Theorem V B and the definition of the quantities $G(n,p)$ in § 54, the number $\mathfrak{G}(n,p)$ is given by

$$(75.3) \qquad (n,p) = \sum_{\sigma=0} G(n,\sigma) - n^2,$$

with the exception of the following combinations:

$$(75.4) \qquad n \text{ arbitrary},\ p=0,\ 1; \quad n=2,\ p=2.$$

As for the affine case, we can also write (75.3) in the form

$$(75.5) \qquad \mathfrak{G}(n,p) = K(n,2)\,K(n+1,p) - nK(n+1,p+1) + n.$$

For the exceptional cases (75.4) we have

$$\mathfrak{G}(n,0)=0, \quad \mathfrak{G}(n,1)=0, \quad \mathfrak{G}(2,2)=1$$

from the results of §§ 70 and 74.

A few of the values of $\mathfrak{A}(n,p)$ and $\mathfrak{G}(n,p)$ may be tabulated as follows:

$\mathfrak{A}(n,p) =$

$n \backslash p$	1	2	3
2	1	8	20
3	15	78	195
4	64	324	900

$\mathfrak{G}(n,p) =$

$n \backslash p$	2	3	4
2	1	2	5
3	3	18	45
4	14	74	200

DEFINITION. *A second fundamental set of affine scalar differential invariants of order* 1 *is a fundamental set of solutions of the differential equations* (71.2); *also a second fundamental set of metric scalar differential invariants of order* 2 *is a fundamental set of solutions of the differential equations* (71.28).

As so defined a second fundamental set of affine scalar differential invariants of order 1 is likewise a first fundamental set of invariants and vice versa; a similar remark applies to the second fundamental set of metric scalar differential invariants of order 2 since for the metric case there are no absolute scalar differential invariants of orders 0 and 1.

DEFINITION. *A second fundamental set of affine or metric scalar differential invariants of order* $p>1$ *or order* $p>2$, *respectively, is a fundamental set of solutions of the differential equations* (69.2 (a)) *or* (69.2 (b)) *such that a sub-set of these solutions, considered as a set of invariants, constitute a second fundamental set of order* $p-1$ *of affine or metric scalar differential invariants, respectively.*

A first and second fundamental set of affine scalar differential invariants of order $p>1$ can never be identical; similarly a first and second funda-

mental set of metric scalar differential invariants cannot be identical when $p > 2$. It is of course evident that the above numbers $\mathfrak{A}(n,p)$ and $\mathfrak{G}(n,p)$ also give the number of invariants in a second fundamental set of affine and metric scalar differential invariants, respectively.

76. Rational invariants

We shall say that any absolute affine scalar differential invariant (69.1a) of order p ($\geqq 1$) is rational if it can be expressed as a rational function of the components $A^{\alpha}_{\beta\gamma\delta_1}; \ldots; A^{\alpha}_{\beta\gamma\delta_1\ldots\delta_p}$; similarly an absolute metric scalar differential invariant (69.1b) of order p ($\geqq 2$) is rational if it is expressible as a rational function of the components $g_{\alpha\beta}; g_{\alpha\beta,\gamma_1\gamma_2}; \ldots; g_{\alpha\beta,\gamma_1\ldots\gamma_p}$. The following theorem will now be proved.

There exists a fundamental set of affine or metric scalar differential invariants of order $p \geqq 1$ or $p \geqq 2$, respectively, which is composed entirely of rational invariants.

The above theorem applies either to the first or second fundamental sets defined in § 75. We shall give an algebraic proof. Considering first the affine case, let us denote the invariants of a first or second fundamental set of order p by

$$G_1,\ G_2, \ldots,\ G_{k-1},\ \ G_k, \ldots,\ G_{\mathfrak{A}}. \tag{76.1}$$

We observe that these invariants are algebraic functions of the components A since they can be regarded as obtained from (67.3) by the algebraic processes of elimination. Let us suppose (1) that the first $k-1$ of the invariants (76.1), where $k \leqq \mathfrak{A}$, are rational, (2) that the remaining invariants $G_k, \ldots, G_{\mathfrak{A}}$ are irrational, and (3) that any algebraic function of the invariants (76.1) which involves at least one invariant of the set $G_k, \ldots, G_{\mathfrak{A}}$ is irrational. In fact if this latter condition were not satisfied the value of the integer k could be increased; we are therefore assuming that k has the greatest possible value, but that since $k \leqq \mathfrak{A}$, there is necessarily at least one irrational invariant in the set (76.1). We shall show that this hypothesis leads to a contradiction.

Since G_α where $\alpha = k, \ldots, \mathfrak{A}$ is algebraic, it must satisfy an irreducible algebraic equation

$$G^m_\alpha + a_{m-1} G^{m-1}_\alpha + \ldots + a_1 G_\alpha + a_0 = 0, \tag{76.2}$$

where the exponent m and the a's depend on the particular invariant G_α; also the a's are rational functions of the components A appearing in (69.1a). Now let $\bar{a}_i$ where $i = 0, \ldots, m-1$ denote the rational function obtained by replacing the A's in a_i by the corresponding components $\bar{A}$. Then it follows from (76.2) and the fact that G_α is a scalar invariant, that

$$G^m_\alpha + \bar{a}_{m-1} G^{m-1}_\alpha + \ldots + \bar{a}_1 G_\alpha + \bar{a}_0 = 0. \tag{76.3}$$

Comparison of (76.2) and (76.3) now shows that $a_i = \bar{a}_i$ or in other words that the a_i are rational invariants. By the above hypothesis, the invariant a_i is therefore expressible as a function of the invariants $G_1, \ldots, G_{k-1}$. Hence (76.2) becomes a relation between the invariants (76.1) which is obviously not satisfied identically. But this is contrary to the fact that the invariants (76.1) are independent as implied by the assumption that these invariants constitute a first fundamental set. The hypothesis that there is necessarily at least one irrational invariant in the set (76.2) has therefore led to a contradiction and the theorem is proved for the case of the affine invariants. A completely analogous proof can evidently be given for the metric case(4).

77. Absolute scalar differential parameters

Consider an absolute affine scalar differential parameter of order (p, q), i.e.

$$G(A^{\alpha}_{\beta\gamma\delta_1}; \ldots; A^{\alpha}_{\beta\gamma\delta_1\ldots\delta_p}; F^{(k)}; F^{(k)}_{\lambda_1}; \ldots; F^{(k)}_{\lambda_1\ldots\lambda_q}), \tag{77.1}$$

where k takes on the values $1, \ldots, w$ (see § 15). It is evident that all differential parameters (77.1) are invariants of the group defined by (67.3) and

$$\left\{\begin{aligned} &\bar{F}^{(k)} = F^{(k)}, \\ &\bar{F}^{(k)}_{\lambda_1} = F^{(k)}_{\mu_1} u^{\mu_1}_{\lambda_1}, \\ &\qquad\cdots \\ &\bar{F}^{(k)}_{\lambda_1\ldots\lambda_q} = F^{(k)}_{\mu_1\ldots\mu_q} u^{\mu_1}_{\lambda_1} \ldots u^{\mu_q}_{\lambda_q} \end{aligned}\right. \tag{77.2}$$

in the variables A, F and the n^2 essential parameters u^{α}_{β} (see § 68).

The symbols of the infinitesimal transformations of the group (77.2) are given by

$$Z^{\mu}_{\nu}(q)f \equiv \sum_{k=1}^{w} \sum_{\sigma=1}^{q} \left\{ \begin{matrix} k\mu \\ \alpha_1 \ldots \alpha_\sigma \nu \end{matrix} \right\} \frac{\partial f}{\partial F^{(k)}_{\alpha_1\ldots\alpha_\sigma}} \tag{77.3}$$

where

$$\left\{ \begin{matrix} k\mu \\ \alpha_1 \ldots \alpha_\sigma \nu \end{matrix} \right\} = F^{(k)}_{\nu\alpha_2\ldots\alpha_\sigma} \delta^{\mu}_{\alpha_1} + \ldots + F^{(k)}_{\alpha_1\ldots\nu} \delta^{\mu}_{\alpha_\sigma} \tag{77.4}$$

Hence we have $X^{\mu}_{\nu}(p)f + Z^{\mu}_{\nu}(q)f$ as the symbols of the infinitesimal transformations of the group (67.3) and (77.2).

Now any differential parameter (77.1) of order (p, q) is an absolute scalar invariant of the group (67.3) and (77.2). However the converse is not necessarily true; this arises from the fact that if $p < q-2$, the extensions of the scalars F appearing in (77.1) may involve derivatives of the Γ's of order greater than p. However, for $p \geqq q-2$ any scalar invariant (77.1) of the group (67.3) and (77.2) is necessarily a differential parameter of order (p, q). We shall therefore assume in the following discussion that the condition $p \geqq q-2$ is satisfied. We shall also exclude the case $(0, 1)$ since the quantities (77.1) are then independent of the affine connection Γ of the space.

The differential equations of the parameter (77.1) are therefore

$$X^{\mu}_{\nu}(p)\,G + Z^{\mu}_{\nu}(q)\,G = 0, \quad (p \geqq q-2), \tag{77.5}$$

and from the results of § 66 these equations form a complete system.

Similarly any absolute metric differential parameter

$$I\,(g_{\alpha\beta};\ \ldots;\ g_{\alpha\beta,\gamma_1\cdots\gamma_p};\ F^{(k)};\ F^{(k)}_{\lambda_1};\ \ldots;\ F^{(k)}_{\lambda_1\cdots\lambda_q}) \tag{77.6}$$

of order (p, q) admits the group defined by (67.4) and (77.2) where $p \geqq q-1$; hence the parameter (77.6) satisfies the complete system

$$Y^{\mu}_{\nu}(p)\,I + Z^{\mu}_{\nu}(q)\,I = 0, \quad (p \geqq q-1), \tag{77.7}$$

where the symbols $Y^{\mu}_{\nu}(p)$ and $Z^{\mu}_{\nu}(q)$ are defined by (68.15) and (77.3) respectively (5).

THEOREM. *A necessary and sufficient condition that the functions* (77.1) *for* $p \geqq q-2$ *and* (77.6) *for* $p \geqq q-1$ *be absolute affine and metric scalar differential parameters of order* (p, q), *respectively, is that the complete systems* (77.5) *and* (77.7) *be satisfied.*

78. INDEPENDENCE OF THE DIFFERENTIAL EQUATIONS OF THE DIFFERENTIAL PARAMETERS

We first consider the independence of the differential equations (77.7). It is obvious that the following theorem is true from the form of these equations.

THEOREM VI. *If the* n^2 *differential equations* (77·7) *are independent for a particular* (n, p, q, w), *they are also independent for* $(n, p+1, q, w)$, $(n, p, q+1, w)$ *and* $(n, p, q, w+1)$.

From the results already established in § 73 and contained in Theorem VB we can say immediately that the equations (77.7) are independent if $n = 2$, $p \geqq 3$ or $n > 2$, $p \geqq 2$. It is found that we must still consider the following three particular cases of the equations (77.7) which we give together with the corresponding number of independent equations:

(a) $n \geqq 2, p = 0, q = 1, w \leqq n-1$ (S independent equations),
(b) $n \geqq 2, p = 0, q = 1, w \geqq n-1$ (n^2 independent equations),
(c) $n \geqq 2, p = 1, q = 2, w = 1$ (n^2 independent equations),

where

$$S = \frac{n(n+1)}{2} + nw - \frac{w(w+1)}{2}. \tag{78.1}$$

When these results have been proved, it will then follow by Theorem VI and the above statement regarding the independence of the equations (77.7) on the basis of Theorem VB that *except for the above case* (a) *the* n^2 *equations* (77.7) *are always independent*. However, we note that the equations for case (a) are all independent when $w = n-1$ since then $S = n^2$ as in fact

demanded by case (*b*). The three particular cases (*a*), (*b*) and (*c*) will now be considered in detail.

Case (*a*). The equations (77.7) become

$$(78.2) \qquad (\delta^{\mu}_{\alpha} g_{\nu\beta} + \delta^{\mu}_{\beta} g_{\alpha\nu}) \frac{\partial I}{\partial g_{\alpha\beta}} + \sum_{k=1}^{w} \delta^{\mu}_{\alpha} F^{(k)}_{\nu} \frac{\partial I}{\partial F^{(k)}_{\alpha}} = 0.$$

Assume

$$(78.3) \qquad g_{\alpha\beta} = \begin{cases} 1 \text{ if } \alpha = \beta, \\ 0 \text{ if } \alpha \neq \beta, \end{cases}$$

and

$$(78.4) \qquad F^{(k)}_{\alpha} = \begin{matrix} 1 \text{ if } k = \alpha & (k = 1, \ldots, w), \\ 0 \text{ if } k \neq \alpha & (k = 1, \ldots, w), \end{matrix}$$

and construct the square matrix from the coefficients of the derivatives in the equations (78.2) which is contained in Table IV. In this table the

TABLE IV

	$\begin{Bmatrix}0\\1\end{Bmatrix}$	$\begin{Bmatrix}1\\2\end{Bmatrix}$	→	$\begin{Bmatrix}w\\w+1\end{Bmatrix}$	$\begin{Bmatrix}w\\w+2\end{Bmatrix}$	→	$\begin{Bmatrix}w\\n\end{Bmatrix}$
(1, 1) (1, 2) ⋮ (1, n)	2 0 … 0 0 1 … 0 ……… 0 0 … 1						
(2, 1) (2, 2) ⋮ (2, n)	☆	1 0 … 0 0 2 … 0 ……… 0 0 … 1					
↓			↘				
(w+1, 1) ⋮ (w+1, w) (w+1, w+1) ⋮ (w+1, n)	☆	☆		1 … 0 0 … 0 …………… 0 … 1 0 … 0 0 … 0 2 … 0 …………… 0 … 0 0 … 1			
(w+2, 1) ⋮ (w+2, w) (w+2, w+2) ⋮ (w+2, n)	☆	☆		☆	1 … 0 0 … 0 …………… 0 … 1 0 … 0 0 … 0 2 … 0 …………… 0 … 0 0 … 1		
↓						↘	
(n, 1) ⋮ (n, w) (n, n)	☆	☆		☆	☆		1 … 0 0 ……… 0 … 1 0 0 … 0 2

element in the row headed by (μ, ν) and column headed by $\begin{pmatrix}\alpha\\\beta\end{pmatrix}$ or $\begin{bmatrix}k\\a\end{bmatrix}$ gives the coefficient of the derivative $\partial I/\partial g_{\alpha\beta}$ or $\partial I/\partial F^{(k)}_{\alpha}$, respectively, in the equations (78.2). The blank sections above the square diagonal sections in this table are composed entirely of zero elements, while the elements in the

sections below the diagonal sections have been left undetermined. For brevity in writing this table we have used the notation

$$\left.\begin{matrix}0\\1\end{matrix}\right\}=\binom{1}{1}\cdots\binom{1}{n},$$

and in general

$$\begin{bmatrix}1\\\sigma\end{bmatrix}\cdots\begin{bmatrix}\tau\\\sigma\end{bmatrix}\binom{\sigma}{\sigma}\cdots\binom{\sigma}{n}.$$

The determinant of the matrix in Table IV which is of order S given by (78.1) has the value 2^n. Consequently there are S independent equations (78.2) *at least*. In other words there are *at most*

$$\frac{n(n+1)}{2}+w+nw-S=\frac{w(w+3)}{2}$$

independent solutions of the equations (78.2). But the $w(w+3)/2$ functions

$$Q^{(ij)}=g^{\sigma\tau}F_{\sigma}^{(i)}F_{\tau}^{(j)},\quad F^{(i)}$$

are functionally independent as can be seen quite readily by letting $F_{\alpha}^{(k)}$ have the particular values given by (78.4), in consequence of which the $Q^{(ij)}$ go over into the $w(w+1)/2$ independent quantities g^{ij}. Hence S gives the actual number of independent equations (78.2).

Case (*b*). Since $S=n^2$ for case (*a*) when $w=n-1$, i.e. all equations (78.2) are independent, it follows by Theorem VI that all equations (77.7) are independent for case (*b*).

Case (*c*). For this case the equations (77.7) become

$$(78.5)\quad (\delta_{\alpha}^{\mu}g_{\nu\beta}+\delta_{\beta}^{\mu}g_{\alpha\nu})\frac{\partial I}{\partial g_{\nu\beta}}+\delta_{\alpha}^{\mu}F_{\nu}^{(1)}\frac{\partial I}{\partial F_{\nu}^{(1)}}+(\delta_{\alpha}^{\mu}F_{\nu\beta}^{(1)}+\delta_{\beta}^{\mu}F_{\alpha\nu}^{(1)})\frac{\partial I}{\partial F_{\alpha\beta}^{(1)}}=0.$$

If $n=2$ we have from case (*a*) and Theorem VI with regard to the order q that the four equations (78.5) are all independent. To establish case (*c*) for $n>2$ we shall prove the following theorem.

THEOREM VII. *If the equations* (78·5) *are all independent for the general metric space of n dimensions, they are all independent for the general metric space of $n+1$ dimensions.*

From our hypothesis there exists a determinant Θ of order n^2, formed from the coefficients of

$$(78.6)\qquad \frac{\partial I}{\partial g_{\alpha\beta}};\quad \frac{\partial I}{\partial F_{\alpha}^{(1)}};\quad \frac{\partial I}{\partial F_{\alpha\beta}^{(1)}}$$

in the equations (78.5) for the general n-dimensional metric space, which does not vanish identically in the independent components $g_{\alpha\beta}$, $F_{\alpha}^{(1)}$, $F_{\alpha\beta}^{(1)}$, i.e.

$$\Theta\neq 0.$$

For the general $(n+1)$-dimensional metric space let us now put

$$g_{\alpha n+1}=g_{n+1\alpha}=\begin{cases}1 \text{ if } \alpha=n+1,\\ 0 \text{ if } \alpha\neq n+1,\end{cases}$$

$$F^{(1)}_{n+1}=0,\quad F^{(1)}_{\alpha n+1}=F^{(1)}_{n+1\alpha}=\begin{cases}1 \text{ if } \alpha=n+1,\\ 0 \text{ if } \alpha\neq n+1,\end{cases}$$

and consider the remaining independent components $g_{\alpha\beta}$, $F^{(1)}_{\alpha}$, and $F^{(1)}_{\alpha\beta}$ to be arbitrary. Then it is possible to form a non-vanishing determinant of order $(n+1)^2$ from the coefficients of the quantities (78.6) in the equations (78.5) for the general metric space of $n+1$ dimensions. This determinant is the determinant of the elements in Table V. The rows of Table V correspond

TABLE V

Θ	0 0 … 0 0 0 … 0 ………… 0 0 … 0		0 0 ⋮ 0
* * … * * * … * ………… * * … *	α	β	0 ⋮ 0
	γ	δ	
* * … *	0………0		2

to a particular pair of values (μ, ν) and the columns to the coefficients of the quantities (78.6). Asterisks are used to denote elements which have been left undetermined. The table is so arranged that in the upper left-hand corner there is the above determinant Θ which does not vanish identically. Consequently the first n^2 rows correspond to values $\mu, \nu=1, \ldots, n$ and the first n^2 columns to coefficients of derivatives (78.6) depending only on subscripts having values $1, \ldots, n$. The remaining rows from top to bottom in Table V are taken to correspond to the values $\mu=n+1$, $\nu=1, \ldots, n$ and $\mu=1, \ldots, n+1$, $\nu=n+1$, while the remaining columns from left to right are taken to correspond to derivatives

$$\frac{\partial I}{\partial g_{in+1}}\quad (i=1, \ldots, n), \quad\text{and}\quad \frac{\partial I}{\partial F^{(1)}_{in+1}}\quad (i=1, \ldots, n+1).$$

With this selection of rows and columns there are four non-vanishing determinants labelled α, β, γ and δ in the lower right-hand part of Table V which are given by

$$\alpha=\begin{vmatrix} g_{11} & \cdots & g_{n1}\\ \\ g_{1n} & \cdots & g_{n}\end{vmatrix},\quad \beta=\begin{vmatrix} F^{(1)}_{11} & \cdots & F^{(1)}_{n1}\\ \\ F^{(1)}_{1n} & \cdots & F^{(1)}_{nn}\end{vmatrix},\quad \gamma=\delta=\begin{vmatrix} 1 & 0 & \ldots & 0\\ 0 & 1 & \ldots & 0\\ \\ 0 & 0 & \ldots & 1\end{vmatrix}$$

The determinant formed from the elements of Table V has the value

$$2\Theta \begin{vmatrix} g_{11}-F^{(1)}_{11} & & g_{n1}-F^{(1)}_{n1} \\ & & \\ g_{1n}-F^{(1)}_{1n} & \cdots & g_{nn}-F^{(1)}_{nn} \end{vmatrix}$$

and so does not vanish identically. Hence all equations (78.5) are independent for the general $(n+1)$-dimensional metric space. This proves Theorem VII and completes the discussion of case (c).

As a consequence of Theorems III A and V A we have the following two statements:

(1) If $n \geqq 3$, there are n^2 independent equations (77.5) for the determination of affine differential parameters (77.1) of order (p, q).

(2) If $n=2$, there are n^2 independent equations (78.5) for the determination of affine differential parameters of order (p, q) where $p \geqq 2$.

We will now prove the following

THEOREM VIII. *The differential equations* (77·5) *for the determination of the affine scalar differential parameters of order* (1, 1) *for the general affinely connected space of two dimensions are independent.*

The differential equations in question are

$$\text{(78.7)} \qquad \begin{pmatrix} \alpha\mu \\ \beta\gamma\delta\nu \end{pmatrix} \frac{\partial G}{\partial A^{\alpha}_{\beta\gamma\delta}} + \sum_{k=1}^{w} \begin{Bmatrix} k\mu \\ \alpha\nu \end{Bmatrix} \frac{\partial G}{\partial F^{(k)}_{\alpha}} = 0, \quad (n=2).$$

Using the notation of (72.2) for the independent components of the normal tensor $A^{\alpha}_{\beta\gamma\delta}$ and letting

$$\text{(78.8)} \qquad F^{(1)}_{1} = \epsilon, \quad F^{(1)}_{2} = \zeta,$$

equations (78.7) for $w=1$ take the following form

$$\text{(78.9)} \qquad \begin{aligned} &\alpha\frac{\partial G}{\partial \alpha}+\beta\frac{\partial G}{\partial \beta} + 2\delta\frac{\partial G}{\partial \delta}+\epsilon\frac{\partial G}{\partial \epsilon}=0, \\ &\delta\frac{\partial G}{\partial \alpha}+\delta\frac{\partial G}{\partial \beta}+(\alpha+\beta)\frac{\partial G}{\partial \gamma} -\epsilon\frac{\partial G}{\partial \zeta}=0, \\ &\gamma\frac{\partial G}{\partial \alpha}+\gamma\frac{\partial G}{\partial \beta} +(\alpha+\beta)\frac{\partial G}{\partial \delta}-\zeta\frac{\partial G}{\partial \epsilon}=0, \\ &\alpha\frac{\partial G}{\partial \alpha}+\beta\frac{\partial G}{\partial \beta} +2\gamma\frac{\partial G}{\partial \gamma} +\zeta\frac{\partial G}{\partial \zeta}=0. \end{aligned}$$

The following determinant of the matrix of this system

$$\begin{vmatrix} 0 & 2\delta & \epsilon & 0 \\ (\alpha+\beta) & 0 & 0 & -\epsilon \\ 0 & (\alpha+\beta) & -\zeta & 0 \\ 2\gamma & 0 & 0 & \zeta \end{vmatrix} \neq 0.$$

Hence the equations (78.9) are all independent. As an immediate consequence of this we have that the equations (78.7) are independent for $w > 1$. This completes the proof of our theorem.

We must also consider the case $p=0$, $q=2$ for which the differential equations are

$$\sum_{k=1}^{w} \begin{Bmatrix} k\mu \\ \alpha\nu \end{Bmatrix} \frac{\partial G}{\partial F^{(k)}_{\alpha}} + \sum_{k=1}^{w} \begin{Bmatrix} k\mu \\ \alpha\beta\nu \end{Bmatrix} \frac{\partial G}{\partial F^{(k)}_{\alpha\beta}} = 0. \tag{78.10}$$

For $n=2$, $w=1$ we may take the independent components

$$F^{(1)}_{1}=\alpha,\quad F^{(1)}_{2}=\beta,\quad F^{(1)}_{11}=\gamma,\quad F^{(1)}_{12}=\delta,\quad F^{(1)}_{22}=\epsilon, \tag{78.11}$$

and obtain the following matrix of the coefficients of equations (78.10):

$$\begin{matrix} \alpha & 0 & 2\gamma & \delta & 0 \\ \beta & 0 & 2\delta & \epsilon & 0 \\ 0 & \alpha & 0 & \gamma & 2\delta \\ 0 & \beta & 0 & \delta & 2\epsilon \end{matrix}$$

The determinant formed from the first, second, third and fifth columns of this matrix does not vanish identically, and therefore the equations (78.10) are independent for $w=1$ and *a fortiori* for $n=2$, $w \geqq 1$.

If $n \geqq 3$, equations (78.10) are not independent if $w=1$. We can however prove that *for $n \geqq 2$, $w \geqq 2$, there are n^2 independent equations* (78.10) *for the determination of the affine scalar differential parameter of order* (0, 2). Since we have shown that they are independent for $n=2$ it is sufficient to show that if equations (78.10) are independent for the general affine space of n dimensions they are independent for the general affine space of $n+1$ dimensions. For $w=2$ this may be done by a simple modification of the discussion for case (c) in the metric case. Since equations (78.10) are essentially of the same form as (78.5) we simply replace $g_{\alpha\beta}$ in the proof in question by $F^{(2)}_{\alpha\beta}$ and the entire discussion remains valid. Having proved the statement for $w=2$, it must also be true for $w>2$.

For $n \geqq 3$, $w=1$ we wish to show that there are $\dfrac{n(n+3)}{2} - 1$ independent equations in the set (78.10). The equations here become the same as those for case (a) for the metric parameters on replacing $F^{(1)}_{\alpha\beta}$ by $g_{\alpha\beta}$ and taking $w=1$. By noting that the $w(w+1)/2$ functions $Q^{(ij)}$ mentioned in the discussion of case (a) can be replaced by the single invariant $F^{\alpha\beta} F^{(1)}_{\alpha} F^{(1)}_{\beta}$, where the $F^{\alpha\beta}$ are formed from the $F^{(1)}_{\alpha\beta}$ in the same way as the $g^{\alpha\beta}$ are formed from the $g_{\alpha\beta}$, we see that the entire discussion given there applies to the present case.

The only remaining singular case arises in the determination of the affine differential parameters of order (1, 0) for $n=2$. The form of the system of equations is the same as that of the equations (74.1). Hence there are three independent equations.

79. Fundamental sets of differential parameters The total number of independent variables A, F in the differential equations (77.5) is easily seen to be

$$\sum_{\alpha=0}^{p} A(n,\alpha)+w\sum_{\alpha=0}^{q} K(n,\alpha),$$

where the $A(n,\alpha)$ and $K(n,\alpha)$ are defined in § 54. Consequently the total number of functionally independent solutions of the equations (77.5) is

$$\begin{aligned}\mathfrak{A}(n,w,p,q)&=\sum_{\alpha=0}^{p} A(n,\alpha)+w\sum_{\alpha=0}^{q} K(n,\alpha)-n^2 \quad (p\geqq q-2),\\ &=n\left[K(n,2)\,K(n+1,p)-K(n+1,p+2)+1\right]+wK(n+1,q),\end{aligned}$$

for $p\geqq 0$, $q\geqq 0$, $w\geqq 1$, $n\geqq 2$, with the exception of the combinations $p=1$, $q=0$, $n=2$ and $p=0$, $q=2$, $n\geqq 3$, $w=1$ for which we have

$$\mathfrak{A}(2,w,1,0)=w+1 \quad (w\geqq 1), \quad \text{and} \quad \mathfrak{A}(n,1,0,2)=2.$$

It is also to be noted that the above formula does not apply to the case of parameters of order (0, 1) which was previously excluded.

Similarly the number of independent components g, F in (77.7), where $k=1, \ldots, w$, is given by

$$\sum_{\alpha=0}^{p} G(n,\alpha)+w\sum_{\alpha=0}^{q} K(n,\alpha).$$

Hence

$$\begin{aligned}\mathfrak{G}(n,w,p,q)&=\sum_{\alpha=0}^{p} G(n,\alpha)+w\sum_{\alpha=0}^{q} K(n,\alpha)-n^2 \quad (p\geqq q-1),\\ &=K(n,2)\,K(n+1,p)-nK(n+1,p+1)+n+wK(n+1,q)\end{aligned}$$

gives the number of functionally independent solutions of the equations (77.7) except for the combinations*

$$\begin{aligned}&\alpha\colon\ n=2,\ p=2,\ q=0,\ w\geqq 1,\\ &\beta\colon\ n\geqq 2,\ p=0,\ q=1,\ w<n-1.\end{aligned}$$

For the combinations α and β the number of independent solutions is given by

$$\begin{aligned}\mathfrak{G}(2,w,2,0)&=\sum_{\alpha=0}^{2} G(2,\alpha)+w-3=w+1,\\ \mathfrak{G}(n,w,0,1)&=G(n,0)+w\sum_{\alpha=0}^{1} K(n,\alpha)-S=\frac{w(w+3)}{2},\end{aligned}$$

respectively.

Definition. *A fundamental set of affine or metric scalar differential parameters of order (p, q) is a set of functionally independent affine or metric scalar differential parameters, each of which is of order (p, q), such that any other one of order (p, q) is expressible as a function of the members of the set.*

* The combination $n\geqq 2$, $p=0$, $q=0$, $w\geqq 1$ need not be considered since there exist no parameters of order (0, 0) which depend on the components $g_{\alpha\beta}$.

It is assumed implicitly in the statement of the above definition that for all the parameters in question, the values of the integers n and w are the same. As thus defined the above fundamental sets of differential parameters correspond to the first fundamental set of scalar invariants of § 75. Extensions of the concept of the second fundamental set of scalar differential invariants to the case of the scalar differential parameters are evidently possible.

It can be shown by a consideration similar to that of § 75 that $\mathfrak{A}(n, w, p, q)$ and $\mathfrak{G}(n, w, p, q)$ give respectively the numbers of affine and metric differential parameters in a fundamental set of order (p, q).

80. Extension to relative tensor differential invariants

Consider a set of variables $\lambda^1, \ldots, \lambda^n$ which transform by the equations

$$\bar{\lambda}^\alpha = |u^a_b|^K \bar{u}^\alpha_\sigma \lambda^\sigma, \tag{80.1}$$

where K is a constant and the quantities $\bar{u}^\alpha_\sigma$ are defined in § 67. It is immediately seen that (80.1) defines a group in the n^2 parameters; also by the method used in § 68 we may show that the parameters u^α_β are essential. Similarly the equations

$$\bar{\mu}_\alpha = |u^a_b|^M u^\sigma_\alpha \mu_\sigma \tag{80.2}$$

define a group in the variables $\mu_1, \ldots, \mu_n$ and the n^2 essential parameters u^α_β. As the inverses of the above transformations (80.1) and (80.2) we have

$$\lambda^\alpha = |u^a_b|^{-K} u^\alpha_\sigma \bar{\lambda}^\sigma, \quad \mu_\alpha = |u^a_b|^{-M} \bar{u}^\sigma_\alpha \bar{\mu}_\sigma,$$

respectively.

As the fundamental differential equations of the group (80.1) we deduce

$$\frac{\partial \bar{\lambda}^\alpha}{\partial u^\sigma_\tau} = (-\delta^\alpha_\nu \bar{\lambda}^\mu + K\delta^\mu_\nu \bar{\lambda}^\alpha)\, \bar{u}^\nu_\sigma \delta^\tau_\mu; \tag{80.3}$$

also

$$\frac{\partial \bar{\mu}_\alpha}{\partial u^\sigma_\tau} = (\delta^\mu_\alpha \bar{\mu}_\nu + M\delta^\mu_\nu \bar{\mu}_\alpha)\, \bar{u}^\nu_\sigma \delta^\tau_\mu \tag{80.4}$$

are the fundamental differential equations of the group (80.2). It follows directly from (80.3) and (80.4) that

$$E^\mu_\nu f \equiv (-\delta^\alpha_\nu \lambda^\mu + K\delta^\mu_\nu \lambda^\alpha) \frac{\partial f}{\partial \lambda^\alpha} + (\delta^\mu_\alpha \mu_\nu + M\delta^\mu_\nu \mu_\alpha) \frac{\partial f}{\partial \mu_\alpha}$$

are the symbols of the infinitesimal transformations of the group (80.1) and (80.2).

Now let T be a relative tensor of weight W with components $T^{\alpha \ldots \beta}_{\gamma \ldots \delta}$ and suppose that there are i subscripts in the set $\gamma, \ldots, \delta$ and j superscripts in the

set $\alpha, \ldots, \beta$. Having recourse to the transformation equations (67.1) we then see that the expression

$$T^{\alpha\ldots\beta}_{\gamma\ldots\delta}\mu_\alpha\ldots\mu_\beta\lambda^\gamma\ldots\lambda^\delta \tag{80.5}$$

is an invariant function of the group composed of (67.1), (80.1) and (80.2) provided that the constants K and M are chosen so that

$$W+iK+jM=0. \tag{80.6}$$

If the tensor T is an affine differential invariant of order $p\,(\geqq 1)$, the expression (80.5) must therefore admit the transformations of the group defined by (67.3), (80.1) and (80.2), it being assumed that (80.6) is satisfied. Hence, we have

$$[X^\mu_\nu(p)+E^\mu_\nu]\,T^{\alpha\ldots\beta}_{\gamma\ldots\delta}\,\mu_\alpha\ldots\mu_\beta\lambda^\gamma\ldots\lambda^\delta=0. \tag{80.7}$$

Similarly the conditions

$$[Y^\mu_\nu(p)+E^\mu_\nu]\,T^{\alpha\ldots\beta}_{\gamma\ldots\delta}\,\mu_\alpha\ldots\mu_\beta\lambda^\gamma\ldots\lambda^\delta=0 \tag{80.8}$$

are satisfied when T is a metric differential invariant of order $p=0$ or $p\geqq 2$. On expansion (80.7) and (80.8) become

$$\begin{aligned} X^\mu_\nu(p)\,T^{\alpha\ldots\beta}_{\gamma\ldots\delta}&=\delta^\mu_\gamma\,T^{\alpha\ldots\beta}_{\nu\ldots\delta}+\ldots+\delta^\mu_\delta\,T^{\alpha\ldots\beta}_{\gamma\ldots\nu}\\ &\quad-\delta^\alpha_\nu\,T^{\mu\ldots\beta}_{\gamma\ldots\delta}-\ldots-\delta^\beta_\nu\,T^{\alpha\ldots\mu}_{\gamma\ldots\delta}+W\delta^\mu_\nu\,T^{\alpha\ldots\beta}_{\gamma\ldots\delta}, \end{aligned} \tag{80.9}$$

and

$$\begin{aligned} Y^\mu_\nu(p)\,T^{\alpha\ldots\beta}_{\gamma\ldots\delta}&=\delta^\mu_\gamma\,T^{\alpha\ldots\beta}_{\nu\ldots\delta}+\ldots+\delta^\mu_\delta\,T^{\alpha\ldots\beta}_{\gamma\ldots\nu}\\ &\quad-\delta^\alpha_\nu\,T^{\mu\ldots\beta}_{\gamma\ldots\delta}-\ldots-\delta^\beta_\nu\,T^{\alpha\ldots\mu}_{\gamma\ldots\delta}+W\delta^\mu_\nu\,T^{\alpha\ldots\beta}_{\gamma\ldots\delta}, \end{aligned} \tag{80.10}$$

respectively, use being made of the fact that (80.6) is satisfied and that the variables λ^α, μ_α are arbitrary. Since the conditions (80.9) and (80.10) are sufficient to insure the fact that T is either an affine or metric tensor differential invariant of order p, we have the following

THEOREM. *A necessary and sufficient condition that a relative tensor T of weight W be an affine differential invariant of order $p\geqq 1$ or a metric differential invariant of order $p=0$, $p\geqq 2$ is that the equations* (80.9) *or* (80.10), *respectively, be satisfied.*

In the case of an affine scalar differential invariant J of the first order and weight W, we have

$$\binom{\alpha\mu}{\beta\gamma\delta\nu}\frac{\partial J}{\partial A^\alpha_{\beta\gamma\delta}}=\delta^\mu_\nu WJ \tag{80.11}$$

in place of (80.9). For $n=2$ we can introduce the independent components $\alpha, \beta, \gamma, \delta$ as in §72 and obtain the following independent equations

$$\begin{aligned} \alpha\frac{\partial\psi}{\partial\alpha}+\beta\frac{\partial\psi}{\partial\beta}\qquad\qquad\qquad\qquad&+2\delta\frac{\partial\psi}{\partial\delta}=1,\\ \delta\frac{\partial\psi}{\partial\alpha}+\delta\frac{\partial\psi}{\partial\beta}+(\alpha+\beta)\frac{\partial\psi}{\partial\gamma}\qquad\qquad&=0,\\ \alpha\frac{\partial\psi}{\partial\alpha}+\beta\frac{\partial\psi}{\partial\beta}\qquad+2\gamma\frac{\partial\psi}{\partial\gamma}\qquad\qquad&=1, \end{aligned} \tag{80.12}$$

where
$$\psi=\frac{\log J}{W}$$

A particular solution of the system (80.12) is $\log(\alpha-\beta)$. Since the most general solution of the homogeneous system associated with (80.12) is an arbitrary function of the expression $(\alpha-\beta)^2/(\alpha\beta-\gamma\delta)$, the most general solution of the complete system (80.12) is

$$\psi=\log(\alpha-\beta)+F\left[\frac{(\alpha-\beta)^2}{\alpha\beta-\gamma\delta}\right],$$

where F denotes an arbitrary function of its argument. Hence the general relative affine scalar invariant of order 1 and weight W for $n=2$ has the form

$$J=(\alpha-\beta)^W F\left[\frac{(\alpha-\beta)^2}{\alpha\beta-\gamma\delta}\right]. \tag{80.13}$$

As particular cases of (80.13) we see that $(\alpha-\beta)$ is an invariant of weight 1, and also that $(\alpha-\beta)^2$ and $(\alpha\beta-\gamma\delta)$ are invariants of weight 2; these results were shown in § 74 by actual calculation.

Similarly, in place of (80.10), we have for a metric scalar differential invariant I of weight W

$$\begin{bmatrix}\mu\\ \alpha\beta\nu\end{bmatrix}\frac{\partial I}{\partial g_{\alpha\beta}}=\delta^{\mu}_{\nu}WI \tag{80.14}$$

for $p=0$. For $n=2$, expansion of (80.14) gives the three independent equations

$$\left\{\begin{aligned} &2\alpha\frac{\partial\psi}{\partial\alpha}+\beta\frac{\partial\psi}{\partial\beta} &&=1,\\ &2\beta\frac{\partial\psi}{\partial\alpha}+\gamma\frac{\partial\psi}{\partial\beta} &&=0,\\ &\alpha\frac{\partial\psi}{\partial\beta}+\beta\frac{\partial\psi}{\partial\gamma} &&=0,\end{aligned}\right. \tag{80.15}$$

where α, β, γ denote components $g_{\alpha\beta}$ as defined in § 74, and where ψ is defined by

$$\psi=\frac{\log I}{W}$$

The most general solution of (80.15) is

$$\psi=\tfrac{1}{2}\log(\alpha\gamma-\beta)^2+\text{const.}$$

Hence

$$I=c\,(\alpha\gamma-\beta^2)^{\frac{W}{2}}=c\,|g_{\alpha\beta}|^{\frac{W}{2}}$$

is the most general metric scalar differential invariant of order zero and weight W for $n=2$.

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(4) See E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces* (B. G. Teubner, 1906), p. 34. The modifications of the proof necessary to show the existence of a fundamental set of rational differential invariants were worked out by J. Levine. See also A. Capelli, *Lezioni sulla Teoria delle Forme Algebriche* (B. Pellerano, 1902), pp. 85–7.

(5) Previous authors in treating the metric case have made the more restrictive condition that $p=q-1$. The restrictions $p \geqq q-2$ and $p \geqq q-1$ for the affine and metric cases, respectively, were not explicitly stated in the above papers by T. Y. Thomas and A. D. Michal, but should be understood to apply there likewise.

CHAPTER VIII

THE EQUIVALENCE PROBLEM

81. Equivalence of generalized spaces

Let $\mathcal{R}$ and $\overline{\mathcal{R}}$ be n-dimensional regions of two n-dimensional generalized spaces $\mathcal{T}$ and $\overline{\mathcal{T}}$ of the same type,* the regions $\mathcal{R}$ and $\overline{\mathcal{R}}$ being covered by coordinate systems x and $\bar{x}$ respectively. Suppose that there exists a one to one reciprocal correspondence of the points of $\mathcal{R}$ and $\overline{\mathcal{R}}$ defined by a set of analytic equations

$$(81.1) \qquad x^\alpha = f^\alpha(\bar{x}^1, \ldots, \bar{x}^n), \quad \bar{x}^\alpha = F^\alpha(x^1, \ldots, x^n).$$

Then it follows from the discussion in § 1 that the functional determinants

$$\frac{\partial f^\alpha}{\partial \bar{x}^\beta} \qquad \frac{\partial F^\alpha}{\partial x^\beta}$$

are not equal to zero at any points $\bar{P}$ and P of the regions   and $\mathcal{R}$, respectively.

Now let $\mathfrak{C}$ denote any geometrical configuration in the region $\mathcal{R}$ and $\bar{\mathfrak{C}}$ the configuration in $\overline{\mathcal{R}}$ into which $\mathfrak{C}$ is transformed by the correspondence (81.1). *We shall say that the spaces $\mathcal{T}$ and $\overline{\mathcal{T}}$ are equivalent throughout the regions $\mathcal{R}$ and $\overline{\mathcal{R}}$ under consideration if the configurations $\mathfrak{C}$ and $\bar{\mathfrak{C}}$ are indistinguishable by means of the intrinsic structure of the spaces $\mathcal{T}$ and $\overline{\mathcal{T}}$, respectively.*

Consider, first, two general affinely connected spaces $\mathcal{T}$ and $\overline{\mathcal{T}}$ and let the configuration $\mathfrak{C}$ consist of a curve C and a family of curves C' intersecting C as illustrated in Fig. 9. Let the curves C and C' be defined by equations of the form

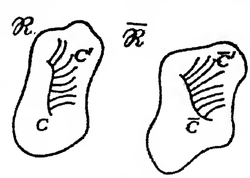

Fig. 9.

$$C: \ x^\alpha = \phi^\alpha(\lambda),$$
$$C': \ x^\alpha = \psi^\alpha(\eta, \lambda),$$

and suppose that the curves C' intersect the curve C for the value $\eta = 0$ of the parameter η. Suppose furthermore that the tangent vectors to the curves C' at points of C are parallel with respect to the curve C; these tangent vectors have components

$$\left(\frac{\partial \psi^\alpha}{\partial \eta}\right)_{\eta=0}.$$

The transformation (81.1) will then carry the curves C and C' into the curves

$$\bar{C}: \ \bar{x}^\alpha = \bar{\phi}^\alpha(\lambda),$$
$$\bar{C}': \ \bar{x}^\alpha = \bar{\psi}^\alpha(\eta, \lambda);$$

* I.e. the spaces $\mathcal{T}$ and $\overline{\mathcal{T}}$ are both affinely connected spaces or both projective spaces, etc.

the tangent vectors to the curves C' at points of C will then have components

$$\left(\frac{\partial \bar{\psi}^{\alpha}}{\partial \eta}\right)_{\eta=0}.$$

Now in consequence of the above assumption that the tangent vectors to the curves C' are parallel with respect to C, we must have

$$\left(\frac{\partial^2 \psi^{\alpha}}{\partial \eta\, \partial \lambda}\right)_{\eta=0} = -L^{\alpha}_{\beta\gamma}(x)\left(\frac{\partial \psi^{\beta}}{\partial \eta}\right)_{\eta=0}\frac{d\phi^{\gamma}}{d\lambda} \tag{81.2}$$

along the curve C. Also if the two affinely connected regions $\mathcal{R}$ and $\bar{\mathcal{R}}$ are to be equivalent under the transformation (81.1), the corresponding equations, i.e.

$$\left(\frac{\partial^2 \bar{\psi}^{\alpha}}{\partial \eta\, \partial \lambda}\right)_{\eta=0} = -\bar{L}^{\alpha}_{\mu\nu}(\bar{x})\left(\frac{\partial \bar{\psi}^{\mu}}{\partial \eta}\right)_{\eta=0}\frac{d\bar{\phi}^{\nu}}{d\lambda}, \tag{81.3}$$

must be satisfied along $\bar{C}$. But

$$\left(\frac{\partial \bar{\psi}^{\alpha}}{\partial \eta}\right)_{\eta=0} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\beta}}\left(\frac{\partial \psi^{\beta}}{\partial \eta}\right)_{\eta=0}, \quad \frac{d\bar{\phi}^{\nu}}{d\lambda} = \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}}\frac{d\phi^{\beta}}{d\lambda},$$

$$\left(\frac{\partial^2 \bar{\psi}^{\alpha}}{\partial \eta\, \partial \lambda}\right)_{\eta=0} = \frac{\partial^2 \bar{x}^{\alpha}}{\partial x^{\beta}\, \partial x^{\gamma}}\frac{d\phi^{\gamma}}{d\lambda}\left(\frac{\partial \psi^{\beta}}{\partial \eta}\right)_{\eta=0} + \frac{\partial \bar{x}^{\alpha}}{\partial x^{\beta}}\left(\frac{\partial^2 \psi^{\beta}}{\partial \eta\, \partial \lambda}\right)_{\eta=0}.$$

Substituting these expressions into the above equations (81.3), we obtain a set of equations which can be reduced to the form

$$\left(\frac{\partial^2 \psi^{\alpha}}{\partial \eta\, \partial \lambda}\right)_{\eta=0} = -\left[\frac{\partial x^{\alpha}}{\partial \bar{x}^{\nu}}\left(\frac{\partial^2 \bar{x}^{\nu}}{\partial x^{\beta}\, \partial x^{\gamma}} + \bar{L}^{\nu}_{\sigma\tau}(\bar{x})\frac{\partial \bar{x}^{\sigma}}{\partial x^{\beta}}\frac{\partial \bar{x}^{\tau}}{\partial x^{\gamma}}\right)\right]\left(\frac{\partial \psi^{\beta}}{\partial \eta}\right)_{\eta=0}\frac{d\phi^{\gamma}}{d\lambda}, \tag{81.4}$$

and which must accordingly be valid along the curve C. Since (81.2) and (81.4) must hold for all such systems of curves C and C', we therefore have

$$L^{\alpha}_{\beta\gamma}(x) = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\nu}}\left[\frac{\partial^2 \bar{x}^{\nu}}{\partial x^{\beta}\, \partial x^{\gamma}} + \bar{L}^{\nu}_{\sigma\tau}(\bar{x})\frac{\partial \bar{x}^{\sigma}}{\partial x^{\beta}}\frac{\partial \bar{x}^{\tau}}{\partial x^{\gamma}}\right]. \tag{81.5}$$

As it is evident that these conditions furnish a sufficient condition for the regions $\mathcal{R}$ and $\bar{\mathcal{R}}$ to be equivalent under (81.1), we obtain the following result.

If $\mathcal{R}$ and $\bar{\mathcal{R}}$ are regions of two general affinely connected spaces $\mathcal{T}$ and $\bar{\mathcal{T}}$, respectively, and if (81.1) *defines a one to one reciprocal correspondence of the points of $\mathcal{R}$ and $\bar{\mathcal{R}}$, a necessary and sufficient condition for the regions $\mathcal{R}$ and $\bar{\mathcal{R}}$ to be equivalent under the transformation* (81.1) *is that the equations* (81.5) *be satisfied.*

The conditions (81.5) which we have deduced for the regions $\mathcal{R}$ and $\bar{\mathcal{R}}$ to be equivalent under the transformation (81.1) might have been inferred of course from the analytical similarity of the present theory with the theory of coordinate transformations of § 9. As above obtained these conditions were deduced directly from the abstract definition of equivalence. We may therefore immediately infer that the regions $\mathcal{R}$ and $\bar{\mathcal{R}}$ of two metric

spaces are equivalent under the transformation (81.1) if and only if the conditions

$$\bar{g}_{\alpha\beta}(\bar{x}) = g_{\mu\nu}(x)\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\beta} \tag{81.6}$$

are satisfied. For the case of the space of distant parallelism, the conditions (81.6) are replaced by the conditions

$$\bar{h}_i^\alpha(\bar{x}) = a_i^k h_k^\beta(x)\frac{\partial \bar{x}^\alpha}{\partial x^\beta}; \tag{81.7}$$

here the constants a_i^k are subject merely to the condition that the determinant $|a_i^k|$ is not equal to zero in the case of the affine space of distant parallelism, and to the conditions (6.8) in the case of the metric space of distant parallelism. Similarly the regions $\mathcal{R}$ and $\bar{\mathcal{R}}$ of two conformal or projective spaces are equivalent under (81.1) if and only if (20.1) or (16.1) are satisfied; analogous conditions apply to the equivalence of two Weyl spaces.

1°. It may be observed that a necessary and sufficient condition for the equivalence of the regions $\mathcal{R}$ and $\bar{\mathcal{R}}$ of two affinely connected spaces of symmetric affine connection is that the paths of $\mathcal{R}$ correspond to the paths of $\bar{\mathcal{R}}$ by (81.1) *without change of parameter*; on the other hand the necessary and sufficient condition for the equivalence of these regions in the case of two projective spaces is merely that paths be transformed into paths, no regard being paid to changes in the parameters of the paths as a result of the transformation (81.1). If the paths of $\mathcal{R}$ and $\bar{\mathcal{R}}$ are transformable by (81.1) without change of parameter, we say that these regions are *affinely equivalent*. When, however, the transformation (81.1) transforms the paths of $\mathcal{R}$ into the paths of $\bar{\mathcal{R}}$ in such a way that parameter changes occur, we say that the two regions are *projectively equivalent*.

These conditions for affine and projective equivalence, which are deducible from the general abstract definition of equivalence, are recognizable as special conditions applying to the particular type of space under consideration. Analogous special conditions of equivalence can be given for the other generalized spaces. For example, it is evidently a necessary and sufficient condition for the regions $\mathcal{R}$ and $\bar{\mathcal{R}}$ to be equivalent in the case of conformal spaces, that angles should be invariant under the transformation (81.1); corresponding to the above terminology we may now say that the regions $\mathcal{R}$ and $\bar{\mathcal{R}}$ of two metric spaces are *conformally equivalent* if the transformation (81.1) leaves angles unaltered, etc.

2°. We can replace the conditions of equivalence given by (81.7) by a set of equations which does not involve the constants a_i^k. We can in fact deduce from (81.7) that

$$\bar{\Delta}^\sigma_{\beta\gamma}\frac{\partial x^\alpha}{\partial \bar{x}^\sigma} = \frac{\partial^2 x^\alpha}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} + \Delta^\alpha_{\mu\nu}\frac{\partial x^\mu}{\partial \bar{x}^\beta}\frac{\partial x^\nu}{\partial \bar{x}^\gamma}, \tag{81.8}$$

where the $\Delta^\alpha_{\beta\gamma}$ are defined in § 6. Conversely it follows from (81.8) that (81.7) is satisfied. To show this we assume that (81.8) is satisfied by some transformation (81.1); then we put

$$\bar{h}_i^\alpha(\bar{x}) = a_i^k(\bar{x})\, h_k^\beta(x)\frac{\partial \bar{x}^\alpha}{\partial x^\beta}, \tag{81.9}$$

and consider that these equations define the quantities $a_i^k(x)$. By differentiation of (81.9) we can then deduce

$$\bar{\Delta}^\sigma_{\beta\gamma}\frac{\partial x^\alpha}{\partial \bar{x}^\sigma} = \frac{\partial^2 x^\alpha}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} + \Delta^\alpha_{\mu\nu}\frac{\partial x^\mu}{\partial \bar{x}^\beta}\frac{\partial x^\nu}{\partial \bar{x}^\gamma} - \frac{\partial x^\sigma}{\partial \bar{x}^\beta} h_l^\alpha\, h_\sigma^i\, b_i^k\frac{\partial a_k^l}{\partial \bar{x}^\gamma}, \tag{81.10}$$

in which the quantities b^i_k are defined as in § 6. From (81.8) the last set of terms in (81.10) must vanish and it follows immediately that the a^k_i are constants; hence (81.9) becomes identical with (81.7) in which the determinant $|a^k_i|$ is different from zero.

It is now evident that (81.6) and (81.8) must furnish the conditions of equivalence in the case of two metric spaces of distant parallelism. This gives us the following result.

If $\mathcal{R}$ and $\overline{\mathcal{R}}$ are regions of two spaces of distant parallelism $\mathcal{T}$ and $\overline{\mathcal{T}}$, respectively, and if (81.1) *defines a one to one reciprocal correspondence of the points of $\mathcal{R}$ and $\overline{\mathcal{R}}$, a necessary and sufficient condition for the regions $\mathcal{R}$ and $\overline{\mathcal{R}}$ to be equivalent under the transformation* (81.1) *is that the equations* (81.8) *be satisfied* (*affine case*) *or that the equations* (81.8) *and* (81.6) *be satisfied* (*metric case*).

82. Normal coordinates and the equivalence problem

Consider two affine spaces of paths $\mathcal{T}$ and $\overline{\mathcal{T}}$, i.e. two affinely connected spaces of symmetric affine connection. Then it is evident that a necessary condition for the existence of two equivalent regions $\mathcal{R}$ and $\overline{\mathcal{R}}$ of these spaces is that a correspondence (81.1) exists such that the sequence of equations

$$\text{(82.1)}\qquad \begin{cases} \bar{A}^{\sigma}_{\beta\gamma\delta}u^{\alpha}_{\sigma} = A^{\alpha}_{\mu\nu\sigma}u^{\mu}_{\beta}u^{\nu}_{\gamma}u^{\sigma}_{\delta}, \\ \bar{A}^{\sigma}_{\beta\gamma\delta\epsilon}u^{\alpha}_{\sigma} = A^{\alpha}_{\mu\nu\sigma\tau}u^{\mu}_{\beta}u^{\nu}_{\gamma}u^{\sigma}_{\delta}u^{\tau}_{\epsilon}, \\ \dots\dots\dots\dots\dots\dots \end{cases}$$

is satisfied, where the A and $\bar{A}$ denote components of normal tensors defined in the regions $\mathcal{R}$ and $\overline{\mathcal{R}}$, respectively, and

$$u^{\alpha}_{\beta} = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\beta}}.$$

Still more particularly, a necessary condition for equivalence is that the above sequence (82.1) possess a *numerical solution*

$$\text{(82.2)}\qquad x^{\alpha} = q^{\alpha},\quad \bar{x}^{\alpha} = \bar{q}^{\alpha},\quad u^{\alpha}_{\beta} = a^{\alpha}_{\beta},$$

such that the determinant of the constants a^{α}_{β}, i.e. the determinant $|a^{\alpha}_{\beta}|$, is not equal to zero. We shall now prove that the existence of the numerical solution (82.2) is sufficient to insure the existence of equivalent regions $\mathcal{R}$ and $\overline{\mathcal{R}}$ of the two spaces(1).

For the above purpose we erect affine normal coordinate systems y and $\bar{y}$ with origins at the points with coordinates $x^{\alpha} = q^{\alpha}$ and $\bar{x}^{\alpha} = \bar{q}^{\alpha}$ of the spaces $\mathcal{T}$ and $\overline{\mathcal{T}}$, respectively. Connect these normal coordinates by the relations

$$\text{(82.3)}\qquad y^{\alpha} = a^{\alpha}_{\beta}\bar{y}^{\beta},$$

where the a^{α}_{β} are the constants in (82.2). Denoting the components of the affine connections in the systems y and $\bar{y}$ by $C^{\alpha}_{\beta\gamma}$ and $\bar{C}^{\alpha}_{\beta\gamma}$, respectively, we have

$$\text{(82.4)}\qquad \begin{cases} C^{\alpha}_{\beta\gamma} = A^{\alpha}_{\beta\gamma\delta}(q)\,y^{\delta} + \dfrac{1}{2!}A^{\alpha}_{\beta\gamma\delta\epsilon}(q)\,y^{\delta}y^{\epsilon} + \dots, \\ \bar{C}^{\alpha}_{\beta\gamma} = \bar{A}^{\alpha}_{\beta\gamma\delta}(\bar{q})\,\bar{y}^{\delta} + \dfrac{1}{2!}\bar{A}^{\alpha}_{\beta\gamma\delta\epsilon}(\bar{q})\,\bar{y}^{\delta}\bar{y}^{\epsilon} + \dots. \end{cases}$$

Since by hypothesis the relations (82.1) hold between the components $A(q)$ and $\bar{A}(\bar{q})$, it follows that the components $C^{\alpha}_{\beta\gamma}$ and $\bar{C}^{\alpha}_{\beta\gamma}$ are related by the equations

$$\bar{C}^{\sigma}_{\beta\gamma} a^{\alpha}_{\sigma} = C^{\alpha}_{\sigma\tau} a^{\sigma}_{\beta} a^{\tau}_{\gamma}, \tag{82.5}$$

i.e. these components transform by the ordinary affine law of transformation when the normal coordinates y^{α} and $\bar{y}^{\alpha}$ undergo the transformation (82.3). The relation (82.3) in conjunction with the relation between the coordinates x^{α}, y^{α} and the coordinates $\bar{x}^{\alpha}$, $\bar{y}^{\alpha}$ defines a one to one reciprocal correspondence (81.1) between the points of two regions $\mathscr{R}$ and $\bar{\mathscr{R}}$ of the given spaces $\mathscr{T}$ and $\bar{\mathscr{T}}$, respectively. Furthermore the correspondence (81.1) so defined is such that

$$\begin{cases} x^{\alpha} = q^{\alpha} \quad \text{when} \quad \bar{x}^{\alpha} = \bar{q}^{\alpha}, \\ \dfrac{\partial x^{\alpha}}{\partial \bar{x}^{\beta}} = a^{\alpha}_{\beta} \quad \text{when} \quad \bar{x}^{\alpha} = \bar{q}^{\alpha} \end{cases} \tag{82.6}$$

This last set of equations can readily be verified by using the fact that at the origin of the normal coordinates the derivatives $\partial x^{\alpha}/\partial y^{\beta}$ are equal to the corresponding δ^{α}_{β}; in fact

$$\left(\frac{\partial x^{\alpha}}{\partial \bar{x}^{\beta}}\right)_{\bar{q}} = \left(\frac{\partial x^{\alpha}}{\partial y^{\sigma}} \frac{\partial y^{\sigma}}{\partial \bar{y}^{\tau}} \frac{\partial \bar{y}^{\tau}}{\partial \bar{x}^{\beta}}\right)_{\bar{q}} = \delta^{\alpha}_{\sigma} a^{\sigma}_{\tau} \delta^{\tau}_{\beta} = a^{\alpha}_{\beta}.$$

It follows from (82.5) and (82.6) that the conditions

$$\bar{\Gamma}^{\sigma}_{\beta\gamma} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\sigma}} = \frac{\partial^2 x^{\alpha}}{\partial \bar{x}^{\beta} \partial \bar{x}^{\gamma}} + \Gamma^{\alpha}_{\sigma\tau} \frac{\partial x^{\sigma}}{\partial \bar{x}^{\beta}} \frac{\partial x^{\tau}}{\partial \bar{x}^{\gamma}}$$

are satisfied by the transformation (81.1) throughout the regions $\mathscr{R}$ and $\bar{\mathscr{R}}$ under consideration.

A necessary and sufficient condition for the existence of equivalent regions $\mathscr{R}$ *and* $\bar{\mathscr{R}}$ *of two affine spaces of paths* $\mathscr{T}$ *and* $\bar{\mathscr{T}}$, *respectively, is that the infinite sequence of equations* (82.1) *possess a numerical solution* (82.2).

The above result can evidently be extended to the case of two general affinely connected spaces $\mathscr{T}$ and $\bar{\mathscr{T}}$ with asymmetric connections, provided that we supplement the sequence (82.1) by the sequence

$$\begin{cases} \bar{\Omega}^{\sigma}_{\beta\gamma} u^{\alpha}_{\sigma} = \Omega^{\alpha}_{\mu\nu} u^{\mu}_{\beta} u^{\nu}_{\gamma}, \\ \bar{\Omega}^{\sigma}_{\beta\gamma,\delta} u^{\alpha}_{\sigma} = \Omega^{\alpha}_{\mu\nu,\sigma} u^{\mu}_{\beta} u^{\nu}_{\gamma} u^{\sigma}_{\delta}, \\ \bar{\Omega}^{\sigma}_{\beta\gamma,\delta\epsilon} u^{\alpha}_{\sigma} = \Omega^{\alpha}_{\mu\nu,\sigma\tau} u^{\mu}_{\beta} u^{\nu}_{\gamma} u^{\sigma}_{\delta} u^{\tau}_{\epsilon}, \end{cases} \tag{82.7}$$

where the $\Omega^{\alpha}_{\beta\gamma}$ are the components of the skew-symmetric part of the affine connection (see § 9). An analogous theorem can likewise be obtained for the case of spaces of distant parallelism, metric spaces and Weyl spaces; the

theorem can also be extended to the case of two projective spaces of paths by having recourse to the $(n+1)$-dimensional representations $A^{\star}_{n+1}$ and $\bar{A}^{\star}_{n+1}$ of these spaces.

83. Complete sets of invariants

We shall say that a set S of differential invariants of a generalized space $\mathcal{T}$ is complete if algebraic necessary and sufficient conditions for the equivalence of a region $\mathcal{R}$ of $\mathcal{T}$ with a region $\bar{\mathcal{R}}$ of a space $\bar{\mathcal{T}}$ of similar type are expressible in terms of the members of the set S.

The theorem of § 82 shows that the totality of normal tensors A constitute a complete set of differential invariants of an affine space of paths; similarly the totality of normal tensors A and tensors Ω occurring in (82.7) form a complete set of differential invariants of the general affinely connected space. Likewise the sets of invariants

$$\text{(I)} \qquad g_{\alpha\beta};\ g_{\alpha\beta,\gamma\delta};\ g_{\alpha\beta,\gamma\delta\epsilon};\ \ldots,$$

$$\text{(II)} \qquad h^i_{j,k};\ h^i_{j,kl};\ h^i_{j,klm};\ \ldots,$$

$$\text{(III)} \qquad {}^{\star}A^{\alpha}_{\beta\gamma\delta};\ {}^{\star}A^{\alpha}_{\beta\gamma\delta\epsilon};\ {}^{\star}A^{\alpha}_{\beta\gamma\delta\epsilon\eta};\ \ldots$$

constitute complete sets of differential invariants of the metric space, the space of distant parallelism, and the projective space of paths, respectively; here the ${}^{\star}A$ are the affine normal tensors in the $(n+1)$-dimensional representation $A^{\star}_{n+1}$ of the projective space.

It can be shown, however, that necessary and sufficient conditions for equivalence can be stated in terms of a finite number of differential invariants; for this purpose we shall need the theorem proved in the following section.

84. A theorem on mixed systems of partial differential equations

Consider the system of differential equations

$$(84.1) \qquad \frac{\partial Z^{\alpha}}{\partial x^{\beta}} = \psi^{\alpha}_{\beta}(Z, x) \quad (\alpha = 1, 2, \ldots, R;\ \ \beta = 1, 2, \ldots, n),$$

in which the Z's are to be considered as functions of the independent variables x^{β} and the ψ^{α}_{β} are analytic functions of their arguments. We seek solutions

$$(84.2) \qquad Z^{\alpha} = Z^{\alpha}(x^1, \ldots, x^n)$$

of (84.1) which satisfy a system of equations

$$(84.3) \qquad F^{(0)}_{\lambda}(Z, x) = 0.$$

It can be shown that the condition for the existence of such a solution is given by the following(2)

THEOREM. *A necessary and sufficient condition that the system of differential equations* (84.1) *admit a solution* (84.2) *satisfying equations* (84.3) *is that there exist an integer* N ($\geqq 1$) *such that the first* N *sets of equations of the sequence*

$$F^{(1)}_\lambda(Z,x)=0;\quad F^{(2)}_\mu(Z,x)=0;\quad F^{(3)}_\nu(Z,x)=0;\quad \dots \tag{84.4}$$

are algebraically consistent considered as equations for the determination of the Z^α *as functions of the independent variables* x^β, *and that all their solutions satisfy the* $(N+1)$*st set of equations in* (84.4), *where* $F^{(1)}_\lambda=0$ *is the set of equations consisting of* (84.3), *the equations of integrability of* (84.1) *and the equations obtained by differentiating* (84.3) *with respect to* x^β *and eliminating the derivatives of* Z^α *which occur by means of* (84.1); *and* $F^{(k+1)}_\tau=0$ *for* $k\geqq 1$ *is the set of equations obtained by differentiating the set of equations* $F^{(k)}_\tau=0$ *with respect to* x^β *and eliminating the derivatives of* Z^α *by means of* (84.1).

By the conditions of integrability of equations (84.1) we mean of course the set of conditions obtained by differentiating (84.1) with respect to x^γ and then eliminating the second derivatives of Z^α by means of

$$\frac{\partial^2 Z^\alpha}{\partial x^\beta \partial x^\gamma}=\frac{\partial^2 Z^\alpha}{\partial x^\gamma \partial x^\beta}$$

and first derivatives of Z^α by means of (84.1).

The necessity condition of the theorem is evident: in case the differential equations (84.1) possess a solution (84.2) satisfying (84.3) it is clear that the number N must exist inasmuch as it is not possible to have an infinite number of independent equations (84.4) in a finite number of variables.

The sufficiency condition of the theorem can be proved quite readily. The first N sets of equations (84.4) can be written in the solved form

$$Z^\alpha=\zeta^\alpha(Z^{M+1},\dots,Z^R;\ x)\quad(\alpha=1,\dots,M\leqq R), \tag{84.5}$$

provided that we assume, which we may do without any loss of generality, that the dependent Z's are those of the set $Z^1,\dots,Z^M$. Differentiation of (84.5) with respect to x^β and elimination of the derivatives of the Z's by (84.1) gives the equations

$$\psi^\alpha_\beta=\frac{\partial\zeta^\alpha}{\partial Z^\sigma}\psi^\sigma_\beta+\frac{\partial\zeta^\alpha}{\partial x^\beta}\quad(\sigma=M+1,\dots,R). \tag{84.6}$$

Let us denote by the symbol $[\phi]$ the expression obtained by eliminating the quantities $Z^1,\dots,Z^M$ in the function $\phi(Z,x)$ by means of (84.5). Thus the function $[\phi]$ depends on the set of arguments $Z^{M+1},\dots,Z^R,x^1,\dots,x^n$. From (84.6) we then obtain

$$[\psi^\alpha_\beta]=\frac{\partial\zeta^\alpha}{\partial Z^\sigma}[\psi^\sigma_\beta]+\frac{\partial\zeta^\alpha}{\partial x^\beta}\quad(\sigma=M+1,\dots,R), \tag{84.7}$$

and this equation is satisfied identically in the arguments $Z^{M+1},\dots,Z^R$, $x^1,\dots,x^n$ as is evident from the method of formation of the sequence (84.4) and the hypothesis that all solutions of the first N sets of these equations,

which are equivalent to (84.5), satisfy the $(N+1)$st set of equations of the sequence (84.4). On account of this it follows that the equations

$$\frac{\partial \zeta^\alpha}{\partial x^\beta} + \frac{\partial \zeta^\alpha}{\partial Z^\sigma}\frac{\partial Z^\sigma}{\partial x^\beta} = [\psi^\alpha_\beta] \quad (\alpha = 1, \ldots, M \leqq R), \tag{84.8}$$

$$\frac{\partial Z^\sigma}{\partial x^\beta} = [\psi^\sigma_\beta] \quad (\sigma = M+1, \ldots, R), \tag{84.9}$$

which are obtained by substituting (84.5) in (84.1), are such that any solution $Z^\sigma = Z^\sigma(x)$ of (84.9) is a solution of (84.8). The integrability conditions of (84.9) are

$$\frac{\partial [\psi^\sigma_\beta]}{\partial x^\gamma} + \frac{\partial [\psi^\sigma_\beta]}{\partial Z^\rho}[\psi^\rho_\gamma] = \frac{\partial [\psi^\sigma_\gamma]}{\partial x^\beta} + \frac{\partial [\psi^\sigma_\gamma]}{\partial Z^\rho}[\psi^\rho_\beta] \quad (\sigma, \rho = M+1, \ldots, R),$$

or

$$\lfloor \partial x^\gamma \qquad \partial Z^\alpha\, \partial x^\gamma \rfloor$$

$$= \left[\frac{\partial \psi^\sigma_\gamma}{\partial x^\beta} + \frac{\partial \psi^\sigma_\gamma}{\partial Z^\alpha}\frac{\partial \zeta^\alpha}{\partial x^\beta}\right] + \left[\frac{\partial \psi^\sigma_\gamma}{\partial Z^\rho} + \frac{\partial \psi^\sigma_\gamma}{\partial Z^\alpha}\frac{\partial \zeta^\alpha}{\partial Z^\rho}\right][\psi^\rho_\beta].$$

By (84.7) these last equations become

$$\left[\frac{\partial \psi^\sigma_\beta}{\partial x^\gamma} + \frac{\partial \psi^\sigma_\beta}{\partial Z^\alpha}\psi^\alpha_\gamma + \frac{\partial \psi^\sigma_\beta}{\partial Z^\rho}\psi^\rho_\gamma\right] = \left[\frac{\partial \psi^\sigma_\gamma}{\partial x^\beta} + \frac{\partial \psi^\sigma_\gamma}{\partial Z^\alpha}\psi^\alpha_\beta + \frac{\partial \psi^\sigma_\gamma}{\partial Z^\rho}\psi^\rho_\beta\right],$$

and hence are satisfied identically in the variables $Z^{M+1}, \ldots, Z^R, x^1, \ldots, x^n$ since they are the result of eliminating $Z^1, \ldots, Z^M$ by (84.5) in certain of the equations of integrability of (84.1). This completes the proof of the theorem.

In the process of obtaining the value of the integer N in the above theorem so as to make N have a least value we observe that each set of equations $F^{(k)}_\eta = 0$ in the sequence (84.4) must contribute at least one additional condition on the quantities Z^α over the conditions imposed on them by all preceding sets. Hence

$$1 \leqq N \leqq R, \tag{84.10}$$

where R is the total number of variables Z^α. By the integer N in the following sections we shall mean the least value of this integer, which will therefore satisfy (84.10).

85. Finite equivalence theorem for affinely connected spaces

Let us write the equations (81.5) in the form

$$\frac{\partial u^\alpha_\beta}{\partial \bar{x}^\gamma} = \bar{L}^\sigma_{\beta\gamma}u^\alpha_\sigma - L^\alpha_{\mu\nu}u^\mu_\beta u^\nu_\gamma, \quad \frac{\partial x^\alpha}{\partial \bar{x}^\beta} = u^\alpha_\beta. \tag{85.1}$$

The differential equations (85.1) are of the form (84.1) as is evident when we denote the $n(n+1)$ variables x^α and u^α_β in (85.1) by $n(n+1)$ variables Z^α

and replace the variables $\bar{x}^\alpha$ in (85.1) by the n variables x^α; there are now no equations of the type (84.3). Corresponding to the sequence of equations (84.4) we now have

$$
(85.2)\qquad
\begin{cases}
\bar{B}^{\sigma}_{\beta\gamma\delta} u^{\alpha}_{\sigma} = B^{\alpha}_{\mu\nu\sigma} u^{\mu}_{\beta} u^{\nu}_{\gamma} u^{\sigma}_{\delta}, \\
\bar{\Omega}^{\sigma}_{\beta\gamma} u^{\alpha}_{\sigma} = \Omega^{\alpha}_{\mu\nu} u^{\mu}_{\beta} u^{\nu}_{\gamma},
\end{cases}
\qquad
\begin{cases}
\bar{B}^{\sigma}_{\beta\gamma\delta,\epsilon} u^{\alpha}_{\sigma} = B^{\alpha}_{\mu\nu\sigma,\tau} u^{\mu}_{\beta} u^{\nu}_{\gamma} u^{\sigma}_{\delta} u^{\tau}_{\epsilon}, \\
\bar{\Omega}^{\sigma}_{\beta\gamma,\delta} u^{\alpha}_{\sigma} = \Omega^{\alpha}_{\mu\nu,\sigma} u^{\mu}_{\beta} u^{\nu}_{\gamma} u^{\sigma}_{\delta},
\end{cases}
$$

where the $B^{\alpha}_{\beta\gamma\delta}$ are the components of the curvature tensor defined by (12.14). The first set of equations (85.2) gives the transformation of the components of the curvature tensor B and the tensor Ω; the second, third, ... sets of equations are the equations of transformation of the successive covariant derivatives of these tensors. The following theorem then results immediately from the theorem of § 84.

THEOREM. *A necessary and sufficient condition for the existence of equivalent regions $\mathscr{R}$ and $\bar{\mathscr{R}}$ of two general affinely connected spaces $\mathscr{T}$ and $\bar{\mathscr{T}}$, respectively, is that there exist an integer $N\ (\geqq 1)$ such that the first N sets of equations* (85.2) *are compatible considered as equations for the determination of x^α and u^α_β as functions of the independent variables $\bar{x}^\alpha$, and that all their solutions*

$$
(85.3)\qquad x^\alpha = f^\alpha(\bar{x}^1, \ldots, \bar{x}^n), \quad u^\alpha_\beta = \phi^\alpha_\beta(\bar{x}^1, \ldots, \bar{x}^n)
$$

satisfy the $(N+1)$st set of equations of the sequence (85.2).

In place of the equations (85.1) we can consider the equivalent system consisting of

$$
(85.4)\qquad \frac{\partial u^\alpha_\beta}{\partial \bar{x}^\gamma} = \bar{\Gamma}^{\sigma}_{\beta\gamma} u^{\alpha}_{\sigma} - \Gamma^{\alpha}_{\mu\nu} u^{\mu}_{\beta} u^{\nu}_{\gamma}, \quad \frac{\partial x^\alpha}{\partial \bar{x}^\beta} = u^\alpha_\beta,
$$

$$
(85.5)\qquad \bar{\Omega}^{\sigma}_{\beta\gamma} u^{\alpha}_{\sigma} = \Omega^{\alpha}_{\mu\nu} u^{\mu}_{\beta} u^{\nu}_{\gamma},
$$

then (85.4) corresponds to the equations (84.1) and (85.5) to the equations (84.3). The resulting theorem of equivalence, which follows in an evident manner from the general existence theorem of § 84, then involves the curvature tensor B determined by the symmetric components $\Gamma^{\alpha}_{\beta\gamma}$ as well as the successive covariant derivatives of the tensors B and Ω, based on the components $\Gamma^{\alpha}_{\beta\gamma}$; the sequence in question can likewise be represented by (85.2).

In the particular case of the affine geometry of paths, i.e. the affinely connected space with symmetric affine connection, the sequence (85.2) reduces to the sequence

$$
(85.6)\qquad
\begin{cases}
\bar{B}^{\sigma}_{\beta\gamma\delta} u^{\alpha}_{\sigma} = B^{\alpha}_{\mu\nu\sigma} u^{\mu}_{\beta} u^{\nu}_{\gamma} u^{\sigma}_{\delta}, \\
\bar{B}^{\sigma}_{\beta\gamma\delta,\epsilon} u^{\alpha}_{\sigma} = B^{\alpha}_{\mu\nu\sigma,\tau} u^{\mu}_{\beta} u^{\nu}_{\gamma} u^{\sigma}_{\delta} u^{\tau}_{\epsilon},
\end{cases}
$$

This latter sequence can in turn be replaced by the sequence (82.1) in the statement of the above equivalence theorem. To see this we consider the first r sets of relations (49.3) in the variables $\bar{A}$, $\bar{B}$ and eliminate the $\bar{B}$ from the right members of (49.3) by means of (85.6); then eliminate the variables B which have been introduced by this process by use of (49.2). There results a set of equations expressing the components $\bar{A}$ of the first r normal tensors in terms of the components A of these tensors and the u^α_β. But these equations must be algebraically equivalent to the first r sets of equations of the sequence (82.1) since (82.1), or an equivalent form derivable from (82.1) by means of the identities (41.1) and (41.2) satisfied by the components of the normal tensors, constitute the only relations which can exist between the quantities A, $\bar{A}$ and u^α_β in question. Conversely we can pass from the r first sets of equations of the sequence (82.1) to the first r sets of equations of the sequence (85.6). This proves the sufficiency condition of the above equivalence theorem when the sequence (82.1) replaces the sequence (85.6), since if this condition is satisfied for the sequence (82.1), it is likewise satisfied for the sequence (85.6) by the result just obtained; the necessity condition of the theorem is of course directly evident.

86. Finite equivalence theorem for metric spaces

In the case of the metric space, we deduce from (81.6) the equations (85.4) in which the $\Gamma^\alpha_{\beta\gamma}$ are Christoffel symbols; these equations then correspond to the equations (84.1) and the equations

$$\bar{g}_{\alpha\beta} = g_{\mu\nu} u^\mu_\alpha u^\nu_\beta, \tag{86.1 a}$$

derived from (81.6), correspond to (84.3). When we differentiate (86.1 a) and eliminate derivatives of the u^α_β by means of (85.4), the resulting equations are satisfied identically in view of (13.12); hence the first set of equations (84.4) consists of (86.1 a) and

$$\bar{B}^\sigma_{\beta\gamma\delta} u^\alpha_\sigma = B^\alpha_{\mu\nu\sigma} u^\mu_\beta u^\nu_\gamma u^\sigma_\delta. \tag{86.1 b}$$

The remaining sets of equations (84.4) correspond to the successive sets of the sequence

$$\left\{ \bar{B}^\sigma_{\beta\gamma\delta,\epsilon} u^\alpha_\sigma = B^\alpha_{\mu\nu\sigma,\tau} u^\mu_\beta u^\nu_\gamma u^\sigma_\delta u^\tau_\epsilon, \right. \tag{86.1 c}$$

Making use of the equations (12.20) and (13.12) we can evidently write the sequence (86.1) in the equivalent form

$$\begin{aligned} \bar{g}_{\alpha\beta} &= g_{\mu\nu} u^\mu_\alpha u^\nu_\beta, \\ \bar{B}_{\alpha\beta\gamma\delta} &= B_{\mu\nu\sigma\tau} u^\mu_\alpha \,..\, u^\tau_\delta, \\ \bar{B}_{\alpha\beta\gamma\delta,\epsilon} &= B_{\mu\nu\sigma\tau,\eta} u^\mu_\alpha \cdots u^\eta_\epsilon \end{aligned} \tag{86.2}$$

By an argument analogous to that given at the end of § 85, we see that the above sequence (86.2) can be replaced by the sequence

$$(86.3)\qquad \begin{aligned} &\bar{g}_{\alpha\beta}=g_{\mu\nu}u^{\mu}_{\alpha}u^{\nu}_{\beta},\\ &\bar{g}_{\alpha\beta,\gamma\delta}=g_{\mu\nu,\sigma\tau}u^{\mu}_{\alpha}\cdots u^{\tau}_{\delta},\\ &\bar{g}_{\alpha\beta,\gamma\delta\epsilon}=g_{\mu\nu,\sigma\tau\eta}u^{\mu}_{\alpha}\cdots u^{\eta}_{\epsilon},\\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

The general existence theorem of § 84 now gives an equivalence theorem for two metric spaces $\mathscr{T}$ and $\bar{\mathscr{T}}$ which is entirely analogous to the theorem stated in § 85, this theorem being based either on the sequence (86.1), (86.2) or (86.3)(3).

In the special case of two metric spaces $\mathscr{T}$ and $\bar{\mathscr{T}}$ of dimensionality n ($\geqq 2$) and of the same constant curvature K, it follows from (50.9) that the second set of equations (86.2) will be satisfied whenever the first set of these equations is satisfied. The conditions for equivalence are therefore satisfied and hence *two spaces of the same constant curvature are always equivalent.*

87. Finite equivalence theorem for spaces of distant parallelism

Write equations (81.8) in the form

$$(87.1)\qquad \frac{\partial u^{\alpha}_{\beta}}{\partial \bar{x}^{\gamma}}=\bar{\Delta}^{\sigma}_{\beta\gamma}u^{\alpha}_{\sigma}-\Delta^{\alpha}_{\mu\nu}u^{\mu}_{\beta}u^{\nu}_{\gamma},\quad \frac{\partial x^{\alpha}}{\partial \bar{x}^{\beta}}=u^{\alpha}_{\beta}.$$

Then, putting

$$\Psi^{\alpha}_{\beta\gamma}=\Delta^{\alpha}_{\beta\gamma}-\Delta^{\alpha}_{\gamma\beta},$$

we have, corresponding to (84.4) in the case of the affine space of distant parallelism, the following sequence

$$(87.2)\qquad \begin{cases} \bar{\Psi}^{\sigma}_{\beta\gamma}u^{\alpha}_{\sigma}=\Psi^{\alpha}_{\mu\nu}u^{\mu}_{\beta}u^{\nu}_{\gamma},\\ \bar{\Psi}^{\sigma}_{\beta\gamma,\delta}u^{\alpha}_{\sigma}=\Psi^{\alpha}_{\mu\nu,\sigma}u^{\mu}_{\beta}u^{\nu}_{\gamma}u^{\sigma}_{\delta},\\ \bar{\Psi}^{\sigma}_{\beta\gamma,\delta,\epsilon}u^{\alpha}_{\sigma}=\Psi^{\alpha}_{\mu\nu,\sigma,\tau}u^{\mu}_{\beta}u^{\nu}_{\gamma}u^{\sigma}_{\delta}u^{\tau}_{\epsilon},\\ \cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

involving the equations of transformation of the components $\Psi^{\alpha}_{\beta\gamma}$ and of the components of the successive covariant derivatives of the tensor Ψ. The appearance of equations corresponding to the equations of transformation of the components of the curvature tensor and its covariant derivatives are lacking in the above sequence (87.2) owing to the vanishing of the curvature tensor in the space of distant parallelism (see § 12).

In the case of the metric space of distant parallelism, the equations (87.1) are to be supplemented by the conditions (86.1a). Now the equations resulting from differentiation of (86.1a) and elimination of the derivatives

of the u^α_β by (87.1) are found to be satisfied identically. Hence the sequence (84.4) becomes

$$
\begin{aligned}
&\bar g_{\alpha\beta} = g_{\mu\nu} u^\mu_\alpha u^\nu_\beta \\
&\overline{\Psi}{}^{\sigma}_{\beta\gamma} u^\alpha_\sigma = \Psi^{\alpha}_{\mu\nu} u^\mu_\beta u^\nu_\gamma, \\
(87.3)\qquad &\overline{\Psi}{}^{\sigma}_{\beta\gamma,\delta} u^\alpha_\sigma = \Psi^{\alpha}_{\mu\nu,\sigma} u^\mu_\beta u^\nu_\gamma u^\sigma_\delta, \\
&\overline{\Psi}{}^{\sigma}_{\beta\gamma,\delta,\epsilon} u^\alpha_\sigma = \Psi^{\alpha}_{\mu\nu,\sigma,\tau} u^\mu_\beta u^\nu_\gamma u^\sigma_\delta u^\tau_\epsilon,
\end{aligned}
$$

for the case under consideration.

The theorem of equivalence of regions $\mathscr{R}$ and $\overline{\mathscr{R}}$ of two affine or metric spaces of distant parallelism can now be stated, as in § 85, on the basis of the sequences (87.2) or (87.3), respectively.

88. Finite equivalence theorem for projective spaces

It follows from the results of § 19 that the problem of the equivalence of two n-dimensional projective spaces of paths is identical with the problem of the equivalence of the $(n+1)$-dimensional affine representations $A^\star_{n+1}$ and $\bar A^\star_{n+1}$ of these spaces (4). Hence the problem of the equivalence of two projective spaces of paths reduces completely to the affine equivalence problem treated in § 85.

89. Equivalence of two dimensional conformal spaces

We shall say that a two dimensional metric space is elliptic or hyperbolic* within the region $\mathscr{R}$ under consideration, according as the determinant , i.e.

$$
(89.1)\qquad \begin{matrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{matrix}
$$

is positive or negative respectively.† It is easily seen that the algebraic sign of the determinant $|g_{\alpha\beta}|$ is not changed by a conformal transformation (7.1) or a transformation of coordinates of the region $\mathscr{R}$; hence the elliptic or hyperbolic character of the space is unaltered by these transformations.

Instead of the equations (20.1) let us take the equations involving the arbitrary function $\sigma(x)$ as given in § 20 as the basic equations in the discussion of the equivalence problem, and let us write these equations in the form

$$
(89.2)\qquad \bar g_{\alpha\beta} d\bar x^\alpha d\bar x^\beta = \sigma(x)\, g_{\alpha\beta} dx^\alpha dx^\beta.
$$

It is desired to find a solution (81.1) of the equations (89.2) for some selection of the function $\sigma(x)$. It is evident from the above remark that a necessary

* It should be observed that the use made here of the terms elliptic and hyperbolic has no relation to the curvature of the space, but refers to the type of the differential equations (89.4).

† Since the determinant $|g_{\alpha\beta}|$ is assumed to be different from zero at any point of the region $\mathscr{R}$, it will not change its algebraic sign within this region.

condition for the existence of such a solution is that the spaces $\mathcal{T}$ and $\bar{\mathcal{T}}$ in question be both elliptic or both hyperbolic; we shall show that this condition is also sufficient.

Under a transformation of coordinates, namely

$$\omega^1 = F^1(x^1, x^2), \quad \omega^2 = F^2(x^1, x^2), \tag{89.3}$$

the contravariant components $g^{\alpha\beta}$ become $\lambda^{\alpha\beta}$ in accordance with the equations

$$\left\{\begin{aligned} \lambda^{11} &= g^{11}\left(\frac{\partial\omega^1}{\partial x^1}\right)^2 + 2g^{12}\frac{\partial\omega^1}{\partial x^1}\frac{\partial\omega^1}{\partial x^2} + g^{22}\left(\frac{\partial\omega^1}{\partial x^2}\right)^2, \\ \lambda^{12} = \lambda^{21} &= g^{11}\frac{\partial\omega^1}{\partial x^1}\frac{\partial\omega^2}{\partial x^1} + g^{12}\left(\frac{\partial\omega^1}{\partial x^2}\frac{\partial\omega^2}{\partial x^1} + \frac{\partial\omega^1}{\partial x^1}\frac{\partial\omega^2}{\partial x^2}\right) + g^{22}\frac{\partial\omega^1}{\partial x^2}\frac{\partial\omega^2}{\partial x^2}, \\ \lambda^{22} &= g^{11}\left(\frac{\partial\omega^2}{\partial x^1}\right)^2 + 2g^{12}\frac{\partial\omega^2}{\partial x^1}\frac{\partial\omega^2}{\partial x^2} + g^{22}\left(\frac{\partial\omega^2}{\partial x^2}\right)^2. \end{aligned}\right. \tag{89.4}$$

Taking first the hyperbolic case, we wish to show that we can determine a coordinate transformation (89.3) such that $\lambda^{11} = \lambda^{22} = 0$. We consider the differential equation

$$g^{11}\left(\frac{\partial F}{\partial x^1}\right)^2 + 2g^{12}\frac{\partial F}{\partial x^1}\frac{\partial F}{\partial x^2} + g^{22}\left(\frac{\partial F}{\partial x^2}\right)^2 = 0. \tag{89.5}$$

If g^{11} and g^{22} both vanish, the coordinates are already in the form that we are seeking. If this is not the case, there is evidently no loss of generality in assuming that the coefficient g^{11} in these equations is not identically zero; in fact if $g^{11} = 0$ and $g^{22} \neq 0$, we can by a mere renumbering of the coordinates cause g^{11} to be different from zero.* It is therefore possible to decompose the equation (89.5) into the two equations

$$\frac{\partial F}{\partial x^1} = \left[-\frac{g^{12}}{g^{11}} + \frac{\sqrt{-G}}{g^{11}}\right]\frac{\partial F}{\partial x^2}, \tag{89.6a}$$

$$\frac{\partial F}{\partial x^1} = \left[-\frac{g^{12}}{g^{11}} - \frac{\sqrt{-G}}{g^{11}}\right]\frac{\partial F}{\partial x^2}, \tag{89.6b}$$

where we have denoted the determinant

$$\begin{vmatrix} g^{11} & g^{12} \\ g^{12} & g^{22} \end{vmatrix} \tag{89.7}$$

by G for brevity. The quantity G is negative for the hyperbolic case under consideration, since the determinants (89.1) and (89.7) have reciprocal values. Hence it is possible to find two (real) independent analytic solutions F^1 and F^2 of (89.5) which are in fact given as solutions of (89.6a) and (89.6b), respectively. Taking the functions F^1 and F^2 so obtained to define the above coordinate transformation (89.3), it follows that the components λ^{11} and λ^{22} in (89.4) vanish identically. The fundamental quadratic differential form with respect to the (ω) coordinate system is therefore given by

$$ds^2 = \alpha(\omega)\, d\omega^1 d\omega^2, \tag{89.8}$$

* Such a renumbering of coordinates is equivalent to a coordinate transformation $x^1 = \bar{x}^2$, $x^2 = \bar{x}^1$.

where α is a function of the coordinates ω^1, ω^2. Similarly the fundamental quadratic differential form for the space $\bar{\mathscr{T}}$ becomes

$$d\bar{s}^2 = \beta(\varpi)\, d\varpi^1 d\varpi^2$$

as the result of a transformation analogous to (89.3).

Suppose now that the function $\beta(\varpi)$ is defined in a region $\bar{\mathscr{R}}$ of the space $\bar{\mathscr{T}}$; then it is evident that the region $\bar{\mathscr{R}}$ can be taken so that $\omega^\alpha = \varpi^\alpha + a^\alpha$, for suitably chosen constants a^α, will be the coordinates of a point of a region $\mathscr{R}$ of $\mathscr{T}$ in which $\alpha(\omega)$ is defined whenever ϖ^α are the coordinates of a point of $\bar{\mathscr{R}}$. Hence

$$\beta(\varpi)\, d\varpi^1 d\varpi^2 = \tau(\omega)\, \alpha(\omega)\, d\omega^1 d\omega^2.$$

will be satisfied by taking $\omega^\alpha = \varpi^\alpha + a^\alpha$ and defining the scalar $\tau(\omega)$ as the ratio $\beta(\omega - a)/\alpha(\omega)$. The correspondence $\omega^\alpha = \varpi^\alpha + a^\alpha$ then defines a correspondence or transformation (81.1) between the regions $\mathscr{R}$ and $\bar{\mathscr{R}}$ such that (89.2) is satisfied for $\sigma(x) = \tau(\omega)$; hence the two regions $\mathscr{R}$ and $\bar{\mathscr{R}}$ are equivalent.

In the elliptic case the determinant (89.1) and hence the determinant (89.7) or G is positive, so that the equations (89.6) involve an imaginary quantity. For the purpose of obtaining a form for the element of distance in the case of an elliptic space, analogous to the form (89.8) for the hyperbolic space, while at the same time dealing only with real quantities, we are led to replace the equations (89.6) by the following equations*

$$(89.9)\qquad \begin{cases} \dfrac{\partial F^1}{\partial x^1} = -\dfrac{g^{12}}{g^{11}}\dfrac{\partial F^1}{\partial x^2} - \dfrac{\sqrt{G}}{g^{11}}\dfrac{\partial F^2}{\partial x^2}, \\[2ex] \dfrac{\partial F^2}{\partial x^1} = \dfrac{\sqrt{G}}{g^{11}}\dfrac{\partial F^1}{\partial x^2} - \dfrac{g^{12}}{g^{11}}\dfrac{\partial F^2}{\partial x^2}. \end{cases}$$

It is then easily seen that the transformation (89.3) which is defined by (89.9) is such that $\lambda^{11} = \lambda^{22}$, $\lambda^{12} = 0$ in (89.4). Hence the fundamental quadratic differential form for the space $\mathscr{T}$ becomes

$$ds^2 = \gamma(\omega)\,[(d\omega^1)^2 + (d\omega^2)^2];$$

introducing analogous coordinates ϖ^α in the space $\bar{\mathscr{T}}$, we have the corresponding form

$$d\bar{s}^2 = \delta(\varpi)\,[(d\varpi^1)^2 + (d\varpi^2)^2]$$

for this latter space. As explained in detail for the case of the hyperbolic space, the above forms for the element of distance now lead to the existence of equivalent regions $\mathscr{R}$ and $\bar{\mathscr{R}}$ of the two elliptic spaces $\mathscr{T}$ and $\bar{\mathscr{T}}$ under consideration.

* The system (89.9) can be put into the form

$$\frac{\partial(F^1 + iF^2)}{\partial x^1} = \left[\frac{-g^{12} + i\sqrt{G}}{g^{11}}\right]\frac{\partial(F^1 + iF^2)}{\partial x^2},$$

where $i = \sqrt{-1}$, and this equation is identical with (89.6a) in the complex variable F. Cf. É. Goursat, *Cours d'Analyse Mathematique*, 4th ed. (Gauthier-Villars, 1927), 3, p. 84.

THEOREM. *If $\mathcal{T}$ and $\overline{\mathcal{T}}$ represent metric spaces of two dimensions, conformally equivalent regions $\mathcal{R}$ and $\overline{\mathcal{R}}$ of the spaces $\mathcal{T}$ and $\overline{\mathcal{T}}$ exist if, and only if, these spaces are both elliptic or both hyperbolic.*

It is evident from the above discussion that in case of the existence of equivalent regions $\mathcal{R}$ and $\overline{\mathcal{R}}$ of the two spaces $\mathcal{T}$ and $\overline{\mathcal{T}}$, the region $\mathcal{R}$ can be taken as a neighbourhood containing an arbitrary point P of $\mathcal{T}$ and the region $\overline{\mathcal{R}}$ as a neighbourhood containing an arbitrary point $\overline{P}$ of the space $\overline{\mathcal{T}}$ (see § 1).

90. Finite equivalence theorem for conformal spaces of three or more dimensions

In considering the equivalence of two conformal spaces of dimensionality $n \geqq 3$, we may take

$$\frac{\partial u_k^i}{\partial \overline{x}^\alpha} = {}^0\overline{\Gamma}^q_{k\alpha} u^i_q - {}^0\Gamma^i_{j\beta} u^j_k u^\beta_\alpha, \quad \frac{\partial x^\alpha}{\partial \overline{x}^\beta} = u^\alpha_\beta \tag{90.1}$$

as the equations corresponding to (84.1); here it is to be observed that we adopt the convention of § 21 regarding the range of indices. Also corresponding to the conditions (84.3) we have

$$\left\{\begin{aligned} &\overline{G}_{\Phi\Psi} = |\, u^\Delta_\Pi \,|^{-2/n} G_{\Delta\Omega} u^\Delta_\Phi u^\Omega_\Psi, \\ &u^\Delta_\infty = u^\Delta_\Phi u^0_\Psi \overline{G}^{\Phi\Psi}, \\ &u^0_\infty = \tfrac{1}{2} u^0_\Phi u^0_\Psi \overline{G}^{\Phi\Psi}, \quad u^\infty_\infty = |\, u^\Delta_\Pi \,|^{2/n}, \\ &\quad = 1, \quad u^\Delta_0 = u^\infty_\Delta = u^\infty_0 = 0. \end{aligned}\right. \tag{90.2a}$$

Suppose, now, that

$$x^\alpha = \phi^\alpha(\overline{x}^0, \overline{x}^1, \ldots, \overline{x}^n), \quad u^i_k = \phi^i_k(\overline{x}^0, \overline{x}^1, \ldots, \overline{x}^n) \tag{90.3}$$

denotes a solution of (90.1) subject to the above conditions (90.2a). We then see immediately that the first set of these equations is in reality of the form

$$x^\Delta = f^\Delta(\overline{x}^1, \ldots, \overline{x}^n), \tag{90.4a}$$

$$x^0 = \overline{x}^0 + \rho(\overline{x}^1, \ldots, \overline{x}^n). \tag{90.4b}$$

Also the function ρ can differ only by an additive constant from the logarithm of the determinant $(x\overline{x})$ formed from (90.4a); in fact we have

$$\frac{\partial \log (x\overline{x})}{\partial \overline{x}^\Delta} = \bar{u}^\Psi_\Phi \frac{\partial u^\Phi_\Psi}{\partial \overline{x}^\Delta}, \tag{90.5}$$

where $\quad \bar{u}^\Psi_\Phi u^\Phi_\Sigma = \delta^\Phi_\Sigma, \quad \bar{u}^\Psi_\Phi u^\Sigma_\Psi = \delta^\Sigma_\Phi.$

Eliminating the derivatives in the right members of (90.5) by means of (90.1), the resulting expression is easily seen to reduce to u^0_Δ. Hence

$$\rho = \log (x\overline{x}) + \text{const.},$$

and (90.3) or (90.4) belongs to the group $^*\mathfrak{G}$. It follows therefore that the above functions u^i_k in the solution of (90.1) and (90.2a) are derivable from

(90.4a) as described in § 21. The problem of the determination of equivalent regions $\mathscr{R}$ and $\overline{\mathscr{R}}$ of two conformal spaces $\mathscr{T}$ and $\overline{\mathscr{T}}$ reduces therefore to the solution of (90.1) under the conditions (90.2a).

When we differentiate (90.2a) with respect to the independent variables $\bar{x}^\alpha$ and then eliminate the resulting derivatives by means of (90.1), the equations so obtained are seen to be satisfied identically. Hence the first set of equations (84.4) consists of (90.2a) and the conditions of integrability of (90.1), the latter being the equations of transformation of the components of the incomplete conformal curvature tensor (see § 22). Taking now, for definiteness, the case where the dimensionality n is an odd integer, let us complete the above equations by the method of § 28 so as to obtain

$$ {}^0\bar{B}^i_{klm} u^p_i = {}^0B^p_{qrs} u^q_k u^r_l u^s_m. \tag{90.2b} $$

The equations (90.2) will be considered to correspond to the first set of equations (84.4); this is evidently legitimate since (90.2) contains all the conditions which correspond strictly to the first set of equations (84.4), and the fact that (90.2) involves necessary conditions in addition to these latter is immaterial from the standpoint of the procedure of § 84. Continuing we construct the sequence

$$ \begin{cases} {}^0\bar{B}^i_{klmu} u^p_i = {}^0B^p_{qrst} u^q_k \dots u^t_u, \\ {}^0\bar{B}^i_{klmuv} u^p_i = {}^0B^p_{qrstw} u^q_k \dots u^w_v, \end{cases} \tag{90.2c} $$

and thus arrive at the following

THEOREM. *A necessary and sufficient condition for the existence of equivalent regions $\mathscr{R}$ and $\overline{\mathscr{R}}$ of two conformal spaces $\mathscr{T}$ and $\overline{\mathscr{T}}$, for which the dimensionality n ($\geqq 3$) is an odd integer, is that there exists an integer N ($\geqq 1$) such that the first N sets of equations* (90.2) *are compatible considered as equations for the determination of x^α and u^i_k as functions of the independent variables $\bar{x}^\alpha$, and that all their solutions* (90.3) *satisfy the $(N+1)$st set of equations of the sequence* (90.2).

It is evident that a similar theorem holds for even values of the dimensionality n ($\geqq 3$) when the above sequence (90.2) is replaced by one involving the components B as defined in § 28 for the value of the dimensionality n in question.

91. Spatial arithmetic invariants

The finite equivalence theorems of the preceding sections give algebraic (i.e. non-differential) conditions for equivalence and hence serve to specify a finite number of differential invariants which constitute a complete set of differential invariants of the space $\mathscr{T}$ in question; the number of these

invariants in the complete set can be described in terms of the integer N occurring in the statement of the equivalence theorem.

By the inequality (84.10) we have that the integer N appearing in the statement of the theorem of § 85 will satisfy the inequalities

$$1 \leqq N \leqq n^2 + n. \tag{91.1}$$

It can be shown quite readily that *the integer N is an arithmetical invariant of the affine space* $\mathfrak{T}$ (5).

For simplicity let us denote the equations of the sequence (85.2) by

$$(B, \Omega) = (\bar{B}, \bar{\Omega}, p), \tag{91.2}$$

where p^α_β now corresponds to the derivative $\partial\bar{x}^\alpha/\partial x^\beta$. In order to prove the invariant character of the integer N we must show that the value of this integer determined by (91.2) is the same as its value when determined by

$$(B, \Omega) = (\bar{\bar{B}}, \bar{\bar{\Omega}}, q), \tag{91.3}$$

where $q^\alpha_\beta = \partial\bar{\bar{x}}^\alpha/\partial x^\beta$ and the quantities $\bar{\bar{B}}$, $\bar{\bar{\Omega}}$ are determined from the components $\bar{\bar{L}}^\alpha_{\beta\gamma}$ which result from the $\bar{L}^\alpha_{\beta\gamma}$ by an arbitrary analytic transformation

$$\bar{x}^\alpha = \psi^\alpha(\bar{\bar{x}}^1, \ldots, \bar{\bar{x}}^n). \tag{91.4}$$

Hence between the quantities $\bar{B}$, $\bar{\Omega}$ and $\bar{\bar{B}}$, $\bar{\bar{\Omega}}$ we have the relations

$$(\bar{B}, \bar{\Omega}) = (\bar{\bar{B}}, \bar{\bar{\Omega}}, r), \quad r^\alpha_\beta = \frac{\partial\bar{\bar{x}}^\alpha}{\partial\bar{x}^\beta}. \tag{91.5}$$

Let

$$x^\alpha = g^\alpha(\bar{x}^1, \ldots, \bar{x}^n), \quad p^\alpha_\beta = h^\alpha_\beta(\bar{x}^1, \ldots, \bar{x}^n) \tag{91.6}$$

denote any solution of the first $N = \bar{N}$ sets of equations (91.2). Then form the equations

$$x^\alpha = \phi^\alpha(\bar{\bar{x}}), \quad q^\alpha_\beta = s^\alpha_\beta(\bar{\bar{x}}), \tag{91.7}$$

where $\phi^\alpha(\bar{\bar{x}})$ is obtained from $g^\alpha(\bar{x})$ by the substitution (91.4), and s^α_β is defined by

$$s^\alpha_\beta = r^\alpha_\sigma p^\sigma_\beta. \tag{91.8}$$

By the substitution (91.5) and (91.8) the equations (91.2) go over into equations (91.3). Hence (91.7) satisfies the first $\bar{N}+1$ sets of equations (91.3). The set S of all solutions of the first $\bar{N}$ sets of equations (91.2) defines a set S' of solutions (91.7) of the first $\bar{N}$ sets of equations (91.3), each of which satisfies the $(\bar{N}+1)$st set of these equations. Now the solutions (91.7) of the set S' constitute all solutions of the first $\bar{N}$ sets of equations (91.3); for if there existed any solution of these equations not in the set S', we could use this solution and the relation (91.4) to define a relation (91.6) by the process by which (91.7) was obtained, and this relation would constitute a solution of the first $\bar{N}$ sets of equations (91.2) not belonging to the set S, which is contrary to hypothesis. Let the integer $N = \bar{\bar{N}}$ be determined by (91.3). We have just shown that $\bar{\bar{N}}$, which is the least value of the integer N deter-

mined by the equations (91.3), cannot be greater than $\bar{N}$. Also $\bar{\bar{N}}$ cannot be less than $\bar{N}$, for it would then follow by an argument similar to the one which we have just made that the integer N determined by the equations (91.2) would be less than $\bar{N}$. Hence $\bar{N} = \bar{\bar{N}}$. This completes the proof that the integer N is an invariant of the affine space $\mathscr{T}$.

As a consequence of the invariant nature of the integer N we can say that the tensors B and Ω, whose components appear in the first $N+1$ sets of equations of the sequence (91.2), constitute a complete set of invariants of the general affinely connected space $\mathscr{T}$.

An analogous discussion suffices to show that the integer N is likewise an arithmetical invariant of the other spaces for which the finite equivalence theorems have been given in the preceding sections; for the projective and conformal spaces the inequality (91.1) will cease to apply but can be replaced by a corresponding inequality determined in an obvious manner.

REFERENCES

(1) H. Vermeil was the first to apply normal coordinates to the equivalence problem in carrying out a suggestion due to F. Klein; see "Bestimmung einer quadratischen Differentialform...", *Math. Ann.* 79 (1919), pp. 289–312. See also T. Y. Thomas and A. D. Michal, ref. (3), Chapter VII.

(2) This theorem is essentially the same as that given by J. E. Wright, *Invariants of Quadratic Differential Forms* (Cambridge Univ. Press, 1908), pp. 15–17. See also T. Levi-Civita, *The Absolute Differential Calculus* (Blackie and Son, 1927), pp. 29–33, and O. Veblen and J. M. Thomas, "Projective invariants of affine geometry of paths", *Ann. of Math.* (2), 27 (1926), pp. 279–96.

(3) E. B. Christoffel, ref. (4), Chapter II, was the first to give an equivalence theorem of this type. He considered however the case where the first N sets of equations equivalent to (86.2) admit a single independent solution, and hence arrived only at sufficient conditions for the equivalence of two quadratic forms.

(4) O. Veblen and J. M. Thomas, ref. (2), gave a projective equivalence theorem which, however, was based on a different sequence than the one described in the text, since they did not use the $(n+1)$-dimensional affine representation. Cf. also T. Y. Thomas, ref. (2), Chapter III, and J. H. C. Whitehead, ref. (3), Chapter III.

(5) See T. Y. Thomas and A. D. Michal, ref. (3), Chapter VII.

CHAPTER IX

REDUCIBILITY OF SPACES

92. Differential conditions of reducibility

If the structure of a generalized space $\mathcal{T}$ is so specialized throughout a region $\mathcal{R}$ of $\mathcal{T}$ that it can be regarded as a component part of the structure of a space $\mathcal{T}^*$ of lesser generality, we will say that the space $\mathcal{T}$ reduces to the space $\mathcal{T}^*$ within the region $\mathcal{R}$. Thus if $\mathcal{T}$ denotes an affine space of paths with components of affine connection $\Gamma^\alpha_{\beta\gamma}$ and if a tensor G exists with symmetric components $g_{\alpha\beta}(x)$ defined throughout a region $\mathcal{R}$ of $\mathcal{T}$, such that the determinant $|g_{\alpha\beta}|$ does not vanish in $\mathcal{R}$ and also such that

(92.1) $$g_{\alpha\beta,\gamma}=0,$$

then $\mathcal{T}$ reduces to a metric space $\mathcal{T}^*$ defined in $\mathcal{R}$ and having

$$ds^2=g_{\alpha\beta}dx^\alpha dx^\beta$$

as its fundamental quadratic differential form. Similarly the existence of the above tensor G and a covariant vector Φ, with components ϕ_α defined in $\mathcal{R}$ such that

(92.2) $$g_{\alpha\beta,\gamma}=g_{\alpha\beta}\phi_\gamma,$$

form the conditions for the affine space of paths $\mathcal{T}$ to reduce to a Weyl space $\mathcal{T}^*$; in fact the equations (92.2) are easily seen to be equivalent to the equations (8.11) which define the components of the affine connection of the Weyl space.

Evidently the condition for a general affinely connected space to reduce to a metric or Weyl space is that the tensor Ω, whose components are the skew-symmetric part of the connection $L^\alpha_{\beta\gamma}$, should vanish, and that the symmetric components of the connection, namely $\Gamma^\alpha_{\beta\gamma}$, be such that equations of the type (92.1) or (92.2), respectively, are satisfied. It follows immediately from (8.5) that a Weyl space $\mathcal{T}$ reduces to a metric space within a region $\mathcal{R}$ of $\mathcal{T}$ if, and only if, the functions ϕ_γ in (92.2) are the gradient of a scalar function.

We cannot speak of the reduction of a projective space of paths to a metric space, nor the reduction of a Weyl space to a conformal space, etc., since the structure of these spaces is of an intrinsically different nature, with the result that the structure of one cannot be regarded as a component part of the structure of the other. Thus in a metric space we may speak of two vectors as being parallel with respect to a curve but in a projective space this concept is without significance. We can, however, corresponding to (92.1) consider the question of the existence of a set of quantities $\Lambda^\alpha_{\beta\gamma}$ which define the paths of a projective space of paths, such that the $\Lambda^\alpha_{\beta\gamma}$ will be Christoffel symbols with respect to a tensor G. The conditions for this are

(92.3) $$\bar{g}_{\alpha\beta,\gamma}=0,$$

where the $\tilde{g}_{\alpha\beta,\gamma}$ are the components of the covariant derivative of G based on the functions $\Lambda^{\alpha}_{\beta\gamma}$ given by (4.4). Writing (92.3) in the form

(92.4) $$g_{\alpha\beta,\gamma}+2g_{\alpha\beta}\phi_{\gamma}+g_{\gamma\beta}\phi_{\alpha}+g_{\alpha\gamma}\phi_{\beta}=0,$$

where $g_{\alpha\beta,\gamma}$ is the covariant derivative of G based on the functions $\Gamma^{\alpha}_{\beta\gamma}$, and multiplying these equations by the contravariant components $g^{\alpha\beta}$, we have

(92.5) $$2(n+1)\phi_{\gamma}=-g^{\alpha\beta}g_{\alpha\beta,\gamma}.$$

Substituting this value of ϕ_{γ} into the left members of (92.4) and denoting the resulting expression by $g_{\alpha\beta\gamma}$, we find

(92.6) $$g_{\alpha\beta\gamma}=g_{\alpha\beta,\gamma}-\frac{g^{\mu\nu}}{n+1}(g_{\alpha\beta}g_{\mu\nu,\gamma}+\tfrac{1}{2}g_{\gamma\beta}g_{\mu\nu,\alpha}+\tfrac{1}{2}g_{\alpha\gamma}g_{\mu\nu,\beta});$$

hence (92.4) takes the form

(92.7) $$g_{\alpha\beta\gamma}=0.$$

The quantities $g_{\alpha\beta\gamma}$ defined by (92.6) may be shown to be the components of a projective-affine tensor (see § 18). As the sufficiency of the above conditions (92.7) can readily be shown, we have the following result: *A necessary and sufficient condition for the existence of a set of components of connection $\Lambda^{\alpha}_{\beta\gamma}$, which are Christoffel symbols with respect to a tensor G within a region $\mathcal{R}$ of a projective space of paths $\mathcal{T}$, is that the Γ's be such that the equations* (92.7) *are satisfied by a set of quantities $g_{\alpha\beta}$ with non-vanishing determinant* $|g_{\alpha\beta}|$. An analogous extension of the equations (92.2) can be made to the case of the projective space of paths(1).

93. FLAT SPACES

A general affinely connected space $\mathcal{T}$ will be said to reduce to a flat space within a region $\mathcal{R}$ of $\mathcal{T}$ if there exists a coordinate system x for the region $\mathcal{R}$ with respect to which the components of the affine connection $L^{\alpha}_{\beta\gamma}(x)$ are equal to zero. Similarly a metric space will be called flat throughout a region $\mathcal{R}$ if a coordinate system exists for this region with respect to which the components of the fundamental metric tensor $g_{\alpha\beta}$ assume constant values. Analogous statements are, of course, to be made for the space of distant parallelism, the conformal and the projective spaces. The following indications express the required transformation conditions for the flatness of the above spaces(2).

(*a*)	$L^{\alpha}_{\beta\gamma}$	(affine space),
(*b*)	$g_{\beta\gamma}\to\pm\delta^{\beta}_{\gamma}$	(metric space),
(*c*)	$h^{\alpha}_{i}\to\pm\delta^{\alpha}_{i}$	(space of distant parallelism),
(*d*)	$G_{\beta\gamma}\to\pm\delta^{\beta}_{\gamma}$	(conformal space),
(*e*)	$\Pi^{\alpha}_{\beta\gamma}\to 0$	(projective space).

Corresponding to the terminology of § 81 we will say that a space of paths is *affinely flat* when the above condition (*a*) is satisfied and *projectively flat* when condition (*e*) is satisfied.

It is seen from the general existence theorem of § 85 that *a necessary and sufficient condition for a general affinely connected space $\mathcal{T}$ to reduce to a flat space throughout a region $\mathcal{R}$ of $\mathcal{T}$ is that the curvature tensor B and the tensor Ω*

vanish identically in some region $\mathcal{R}^*$ *containing* $\mathcal{R}$.* Similarly it follows from the theorems of § 86 and § 87 that *a necessary and sufficient condition for a metric space or a space of distant parallelism to be flat throughout a region* $\mathcal{R}$ *of the space is that the curvature tensor* B *or the scalars* $h^i_{j,k}$, *respectively, be equal to zero identically.*

From the results of § 89 it follows that a two dimensional metric space is always conformally flat. The theorem of § 90 shows us that for $n \geqq 3$, *a necessary and sufficient condition that a metric space be conformally flat throughout a region* $\mathcal{R}$ *is that the components* ${}^0B^i_{j\alpha\beta}$ *of the incomplete conformal curvature tensor should vanish identically.* Likewise, *the identical vanishing of the projective curvature tensor* ${}^\star B$ *is a necessary and sufficient condition for a projective space of paths to reduce to a flat space.*†

In the case that the space $\mathcal{T}$ is a metric space we shall show that the above theorem can be put in the form: *a necessary and sufficient condition that a metric space be projectively flat is that it should be a space of constant curvature.* To see this we first observe that for a metric space $B_{jk}=B_{kj}$, as may be seen from (49.9) by making use of (41.16). Hence by evaluation of (18.4) we have

$$W^m_{jkl}=B^m_{jkl}+\frac{1}{n-1}(\delta^m_k B_{jl}-\delta^m_l B_{jk}).$$

On multiplying by g_{im} and summing on the index m, we see that the conditions $W^i_{jkl}=0$ give us for $n \geqq 3$ the equations

$$B_{ijkl}=\frac{1}{n-1}(g_{il}B_{jk}-g_{ik}B_{jl}). \tag{93.1}$$

On multiplying (93.1) by g^{jk} and summing on the repeated indices, we find

$$B_{il}=\frac{1}{n-1}(g_{il}B-B_{il}),$$

or

$$B_{il}=\frac{B}{n}g_{il}. \tag{93.2}$$

Hence (93.1) becomes

$$B_{ijkl}=\frac{B}{n(n-1)}(g_{il}g_{jk}-g_{ik}g_{jl}). \tag{93.3}$$

But we saw in § 50 that this is precisely the condition that $\mathcal{T}$ be a space of constant curvature. Conversely, if (93.3) is satisfied $W^i_{jkl}=0$.

For $n=2$ the equations (93.2) are satisfied identically. To show that in this case also we must have $B=\text{const.}$ we use the conditions ${}^\star B^0_{jkl}=0$, which we see from (18.7) reduce to

$$B_{jk,l}-B_{jl,k}=0,$$

* It can be shown that the regions $\mathcal{R}$ and $\mathcal{R}^*$ can be taken to be identical. A similar remark can be made with reference to each of the above cases (*b*), ..., (*e*).

† For $n \geqq 4$ it follows from equations (22.7) that the vanishing of all the components ${}^0B^i_{j\alpha\beta}$ is a consequence of the identical vanishing of $Y^\Omega_{\Lambda\Delta\Phi}$. Similarly, from (18.6) it follows that for $n \geqq 3$ the conditions ${}^\star B^0_{jkl}=0$ are a consequence of $W^i_{jkl}=0$.

or
$$g_{jk}\frac{\partial B}{\partial x^l}-g_{jl}\frac{\partial B}{\partial x^k}=0,$$
on making use of the fact that $g_{ij,k}=0$. If we multiply these equations by g^{jk} and sum on the repeated indices, we see that
$$\frac{\partial B}{\partial x^l}=0,$$
i.e. B is a constant. The converse is obvious. Hence our theorem is proved.

94. Reducibility of the general affinely connected space to a space of distant parallelism

A general affinely connected space will reduce to an affine space of distant parallelism if we can find n independent solutions h_i^α of the equations

$$\frac{\partial h^\alpha}{\partial x^\gamma}+h^\beta L^\alpha_{\beta\gamma}=0. \tag{94.1}$$

The integrability conditions of these equations are

$$h^\beta B^\alpha_{\beta\gamma\delta}=0, \tag{94.2}$$

where the tensor B is the curvature tensor formed from the components $L^\alpha_{\beta\gamma}$. Equations (94.2) will have n independent solutions if and only if

$$B^\alpha_{\beta\gamma\delta}=0. \tag{94.3}$$

Conversely, if (94.3) is satisfied, (94.1) is completely integrable and will therefore have n independent solutions (cf. § 13). Hence we have the

Theorem. *A necessary and sufficient condition for a general affinely connected space $\mathscr{T}$ to reduce to an affine space of distant parallelism throughout a region $\mathscr{R}$ of $\mathscr{T}$ is that the curvature tensor B formed from the components $L^\alpha_{\beta\gamma}$ should vanish.*

95. Algebraic conditions for the reducibility of the affine space of paths to a metric space

Let us consider the expanded form of the equations (92.1), namely

$$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}=g_{\sigma\beta}\Gamma^\sigma_{\alpha\gamma}+g_{\alpha\sigma}\Gamma^\sigma_{\beta\gamma}, \tag{95.1}$$

and apply the general existence theorem of § 84. This leads to the following sequence.*

$$\begin{aligned} &g_{\alpha\beta}=g_{\beta\alpha},\\ &g_{\sigma\beta}B^\sigma_{\alpha\gamma\delta}+g_{\alpha\sigma}B^\sigma_{\beta\gamma\delta}=0,\\ &g_{\sigma\beta}B^\sigma_{\alpha\gamma\delta,\epsilon}+g_{\alpha\sigma}B^\sigma_{\beta\gamma\delta,\epsilon}=0,\end{aligned} \tag{95.2}$$

* The general set of equations of the sequence (95.2) can be obtained from the preceding set by covariant differentiation. The method of § 84 involving partial differentiation for the formation of the successive equations of the sequence (84.4) gives rise to equations in the form (95.2) by the addition of suitable expressions which vanish in view of the preceding relations.

In order that a solution $g_{\alpha\beta}$ of equations (95.1) may be taken as the components of the fundamental tensor of a metric space, it is necessary that the determinant $|g_{\alpha\beta}|$ shall not vanish in the region $\mathscr{R}$. It is therefore necessary that one of the solutions $\overset{(1)}{g}_{\alpha\beta}$ of the first N sets of the sequence (95.2) be such that the determinant $|\overset{(1)}{g}_{\alpha\beta}| \neq 0$. By taking linear combinations of the solution $\overset{(1)}{g}_{\alpha\beta}$ and other solutions of the first N sets of (95.2) we can evidently obtain a fundamental system of solutions $\overset{(i)}{g}_{\alpha\beta}$ of these equations, where $i=1, \ldots, p$, such that none of the determinants $|\overset{(i)}{g}_{\alpha\beta}|$ vanish identically. Assuming now that all the $\overset{(i)}{g}_{\alpha\beta}$ satisfy the $(N+1)$st set of equations (95.2), we obtain a solution $g_{\alpha\beta}$ of (95.1) by integration of a completely integrable system corresponding to (84.9). Hence the solution $g_{\alpha\beta}(x)$ of (95.1) can be made to take on at an arbitrary point P initial values corresponding to any one of the p fundamental solutions $\overset{(i)}{g}_{\alpha\beta}$; the determinant $|g_{\alpha\beta}|$ will therefore not vanish in the neighbourhood of the point P. This gives us the following(3)

THEOREM. *A necessary and sufficient condition for an affine space of paths $\mathscr{T}$ to reduce to a metric space throughout a region $\mathscr{R}$ of $\mathscr{T}$ is that there exist an integer N ($\geqq 1$) such that the first N sets of equations* (95.2) *admit a fundamental system of solutions $\overset{(i)}{g}_{\alpha\beta}$, where $i=1, \ldots, p$, with non-vanishing determinants $|\overset{(i)}{g}_{\alpha\beta}|$, such that each solution $\overset{(i)}{g}_{\alpha\beta}$ satisfies the $(N+1)$st set of equations* (95.2).

By considering the solutions $a_{\alpha\ldots\beta}(x)$ of the equations

(95.3) $$a_{\alpha\ldots\beta,\gamma}=0,$$

we arrive at an obvious generalization of the above existence theorem. Now it is readily shown that the conditions (95.3) are sufficient for the existence of the homogeneous first integral

(95.4) $$a_{\alpha\ldots\beta}\frac{dx^\alpha}{ds}\cdots\frac{dx^\beta}{ds}=\text{const.}$$

of the equations of the paths (3.1). Hence the theorem of § 84 gives us directly an algebraic sufficient condition for the existence of a first integral (95.4).

96. ALGEBRAIC CONDITIONS FOR THE REDUCIBILITY OF THE AFFINE SPACE OF PATHS TO A WEYL SPACE

The theorem of § 95 can readily be extended to give algebraic conditions for the reduction of an affine space of paths to a Weyl space. Writing (92.2) in the expanded form

(96.1) $$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}=g_{\sigma\beta}\Gamma^\sigma_{\alpha\gamma}+g_{\alpha\sigma}\Gamma^\sigma_{\beta\gamma}+g_{\alpha\beta}\phi_\gamma,$$

we have as the conditions of integrability

(96.2) $$g_{\sigma\beta}B^\sigma_{\alpha\gamma\delta}+g_{\alpha\sigma}B^\sigma_{\beta\gamma\delta}+g_{\alpha\beta}(\phi_{\gamma,\delta}-\phi_{\delta,\gamma})=0.$$

Assuming the determinant $|g_{\alpha\beta}|\neq 0$, we multiply (96.2) by $g^{\alpha\beta}$ and sum on the indices α, β so as to obtain

$$\phi_{\gamma,\delta}-\phi_{\delta,\gamma}=\frac{2}{n}B^\sigma_{\sigma\delta\gamma}.$$

When we eliminate the left members of these latter equations from (96.2) we have a system of equations which can be put into the form

$$g_{\alpha\beta}\, G^{\alpha\beta}_{\mu\nu\sigma\tau} = 0, \quad g_{\alpha\beta} = g_{\beta\alpha}; \tag{96.3a}$$

these equations correspond to the first set of conditions (84.4). By successive differentiation of (96.3a) and elimination of the derivatives of $g_{\alpha\beta}$ by means of (96.1), we obtain a sequence of the form

$$\begin{cases} g_{\alpha\beta}\, G^{\alpha\beta}_{\mu\nu\sigma\tau\epsilon} = 0, \\ g_{\alpha\beta}\, G^{\alpha\beta}_{\mu\nu\sigma\tau\epsilon\eta} = 0, \\ \cdots \end{cases} \tag{96.3b}$$

Then, since the condition that the determinant $|g_{\alpha\beta}|$ be different from zero is a necessary condition for the given affine space of paths to reduce to a Weyl space, for which the $g_{\alpha\beta}$ are the components of the fundamental metric tensor, we are led to the following(4)

THEOREM. *A necessary and sufficient condition for an affine space of paths $\mathscr{T}$ to reduce to a Weyl space throughout a region $\mathscr{R}$ of $\mathscr{T}$ is that there exist an integer N ($\geqq 1$) such that the first N sets of equations* (96.3) *admit a fundamental system of solutions $g^{(i)}_{\alpha\beta}(x)$, where $i = 1, \ldots, p$, with non-vanishing determinants $|g^{(i)}_{\alpha\beta}|$, such that each solution $g^{(i)}_{\alpha\beta}$ satisfies the $(N+1)$st set of equations* (96.3).

REFERENCES

(1) See T. Y. Thomas, "Note on the projective geometry of paths", *Bull. Amer. Math. Soc.* **31** (1925), pp. 318–22. Cf. also J. M. Thomas, "First integrals in the geometry of paths", *Proc. N.A.S.* **12** (1926), pp. 117–24.

(2) B. Riemann, ref. (7), Chapter VI, p. 403, gave the vanishing of the curvature tensor as a necessary condition for a Riemann space to be flat. R. Lipschitz and G. Ricci showed that this condition of Riemann was also sufficient. See R. Lipschitz, "Untersuchungen in Betreff der ganzen homogenen Functionen von n Differentialen", *Journ. für reine und ange. Math.* **70** (1869), pp. 71–102, and G. Ricci, "Principii di una teoria delle forme differenziale quadratiche", *Ann. di Mat.* (2), **12** (1884), pp. 135–67. For $n=3$ a necessary and sufficient condition for conformal flatness was given by É. Cotton, ref. (4), Chapter IV. H. Weyl, ref. (2), Chapter I, showed that for $n>3$ the vanishing of the conformal-affine curvature tensor was a necessary condition for conformal flatness. A. Finzi derived conditions equivalent to ${}^0B^i_{j\alpha\beta}=0$ as sufficient conditions for conformal flatness; see "Sulla rappresentabilità conforme di due varietà ad n dimensioni l' una sull' altra", *Atti del R. Istituto Veneto*, **80**^{2} (1921), pp. 777–89. J. A. Schouten showed that Finzi's conditions were also necessary, and also showed that for $n>3$ the one set of conditions was a consequence of the other; see "Ueber die konforme Abbildungen...", *Math. Zeit.* **11** (1921), pp. 58–88. H. Weyl, ref. (6), Chapter I, gave necessary and sufficient conditions for the flatness of affine and projective spaces, as well as for conformal flatness.

(3) L. P. Eisenhart and O. Veblen, ref. (3), Chapter I, gave sufficient conditions for the reducibility of an affine space to a metric space. O. Veblen and T. Y. Thomas, ref. (4), Chapter V, gave necessary and sufficient conditions without, however, making use of the general existence theorem of § 84.

(4) See J. M. Thomas, ref. (1).

CHAPTER X

FUNCTIONAL ARBITRARINESS OF SPATIAL INVARIANTS

In this chapter we consider the question of the determination in finite form of the functional arbitrariness of sets of quantities $A^{\alpha}_{\beta\gamma\delta}$ and $g_{\alpha\beta,\gamma\delta}$, consistent with the existence of an affine and metric space for which these quantities will be the components of the first normal tensor and second extension of the fundamental metric tensor, respectively. An approach to this problem is already to be found in § 46 where, however, the required conditions on the A's and g's are given in terms of an infinite sequence of conditions. As this question is essentially one in the theory of systems of partial differential equations, we begin by deriving a *regular form* for such systems which will have direct application to the present problem (1).

97. Regular systems of partial differential equations

Consider a system of L partial differential equations, linear and of the first order in w dependent variables $v_1, \ldots, v_w$ and n independent variables $x^1, \ldots, x^n$, namely

$$(97.1) \qquad \sum_{k=1}^{w} \sum_{\alpha=1}^{n} a^{\alpha}_{ik} \frac{\partial v_k}{\partial x^{\alpha}} + c_i = 0 \quad (i = 1, \ldots, L).$$

The coefficients a^{α}_{ik} and c_i are functions of x^{α} and v_k. It is assumed also that the left members of (97.1) are linearly independent in the derivatives of the dependent variables v_k.

Let us suppose that there are $L_1 \leqq L$ equations (97.1) which are independent in the derivatives $\partial v_k/\partial x^1$ and, as the integer L_1 is conceivably dependent on the coordinate system x, let us suppose that coordinates x^{α} have been selected for which L_1 will have its maximum value. We can then divide our equations into two sets: a set S_1 consisting of L_1 equations which can be solved for L_1 of the derivatives $\partial v_k/\partial x^1$ and a set S_2 from which all derivatives $\partial v_k/\partial x^1$ can be eliminated. Now suppose that the set of equations S_2 contains L_2 equations independent in the derivatives $\partial v_k/\partial x^2$ and in fact that coordinates x^{α} are selected so that L_2 has its maximum possible value, under the restriction that the above integer L_1 is unchanged. This makes it possible to divide the set S_2 into two sets $S_2^{\star}$ and $S_3^{\star}$ such that $S_2^{\star}$, consisting of L_2 equations, can be solved for L_2 of the derivatives $\partial v_k/\partial x^2$, and such that the derivatives $\partial v_k/\partial x^2$ can be eliminated entirely from the set $S_3^{\star}$. Proceeding in this way we arrive at a coordinate system (which is obviously one of an

infinity of such coordinate systems) for which our system of equations (97.1) can be put into the form

$$\sum_{k=1}^{w}\sum_{\alpha=1}^{n} b^{\alpha}_{ik\beta}\frac{\partial v_k}{\partial x^{\alpha}}+c_{i\beta}=0 \quad \begin{pmatrix}\beta=1,\ldots,n\\ i=1,\ldots,L_{\beta}\end{pmatrix}, \tag{97.2}$$

where $b^{\alpha}_{ik\beta}=0$ if $\alpha<\beta$. A system of coordinates x^{α} with respect to which (97.1) can be put into the form (97.2), in which the integers L_{β} are characterized by the abovementioned property, is said to be *non-singular*; otherwise the coordinate system is said to be *singular*.

If (97.1) is written in the form

$$\sum_{k=1}^{w}\sum_{\alpha=1}^{n} b^{\alpha}_{ik\beta}\frac{\partial v_k}{\partial x^{\alpha}}+c_{i\beta}=0 \quad \begin{pmatrix}\beta=1,\ldots,n\\ i=1,\ldots,J_{\beta}\end{pmatrix},$$

where $b^{\alpha}_{ik\beta}=0$ if $\alpha<\beta$, with respect to a singular coordinate system, then $J_1<L_1$; or if $J_i=L_i$ for $i=1,\ldots,r$, then $J_{r+1}<L_{r+1}$. Obviously the inequality $r\leqq n-1$ is here satisfied.

Now assume a non-singular choice of independent variables x^{α} and make the transformation

$$\bar{x}^{\sigma}=x^{\sigma}+m^{\sigma}_{\tau}x^{\tau}, \tag{97.3}$$

where the m^{σ}_{τ} are constants. If the dependent variables v_k transform as *scalars*, the law of transformation of their derivatives is given by

$$\frac{\partial v_k}{\partial x^{\alpha}}=\frac{\partial v_k}{\partial \bar{x}^{\alpha}}+m^{\sigma}_{\alpha}\frac{\partial v_k}{\partial \bar{x}^{\sigma}},$$

and hence equations (97.2) become

$$\sum_{k=1}\sum_{\alpha=1}(b^{\alpha}_{ik\beta}+m^{\alpha}_{\sigma}b^{\sigma}_{ik\beta})\frac{\partial v_k}{\partial \bar{x}^{\alpha}}+c_{i\beta}=0. \tag{97.4}$$

For a fixed value α belonging to the set $1,\ldots,n-1$ assume that $m^{\sigma}_{\tau}=0$ except when $\sigma=\alpha$. Then if the constants m^{α}_{τ} are sufficiently small equations (97.4) for $\beta=\alpha$ must be linearly independent with respect to the derivatives $\partial v_k/\partial\bar{x}^{\alpha}$. For the above choice of the constants m^{σ}_{τ} the coefficients of the derivatives $\partial v_k/\partial\bar{x}^{\gamma}$, for $\gamma=1,\ldots,\alpha-1$, in the equations (97.4) will be equal to the coefficients of the corresponding derivatives $\partial v_k/\partial x^{\gamma}$ in the system (97.2). Hence the set of forms

$$\sum_{k=1}^{w} m^{\alpha}_{\sigma}b^{\sigma}_{ik\beta}\frac{\partial v_k}{\partial \bar{x}^{\alpha}} \quad (\beta>\alpha,\ \alpha \text{ not summed})$$

will be linearly dependent on the forms

$$\sum_{k=1}^{w}(b^{\alpha}_{ik\alpha}+m^{\alpha}_{\sigma}b^{\sigma}_{ik\alpha})\frac{\partial v_k}{\partial \bar{x}^{\alpha}} \quad (\alpha \text{ not summed}),$$

since otherwise the original choice of variables x^{α} would be singular contrary to hypothesis. Or

$$m^{\alpha}_{\sigma}b^{\sigma}_{ik\beta}=\sum_{j=1}^{L_{\alpha}}\lambda^{\star\alpha}_{ij\beta}(b^{\alpha}_{jk\alpha}+m^{\alpha}_{\sigma}b^{\sigma}_{jk\alpha}) \quad (\beta>\alpha,\ \alpha \text{ not summed}). \tag{97.5}$$

Now take $m^\alpha_\beta = m$, and $m^\sigma_\tau = 0$ otherwise. Then (97.5) gives

$$mb^\beta_{ik\beta} = \sum_{j=1}^{L_\alpha} \lambda^{*\alpha}_{ij\beta}(b^\alpha_{jk\alpha} + mb^\beta_{jk\alpha}) \quad (\beta > \alpha,\ \alpha \text{ and } \beta \text{ not summed}).$$

Hence, if we let m approach zero, we have

$$b^\beta_{ik\beta} = \sum_{j=1}^{L_\alpha} \lambda^\alpha_{ij\beta} b^\alpha_{jk\alpha} \quad \begin{pmatrix}\beta > \alpha,\ i = 1, \ldots, L_\beta \\ \alpha,\ \beta \text{ not summed}\end{pmatrix}, \tag{97.6}$$

where

$$\lambda^\alpha_{ij\beta} = \lim_{m \to 0} \left[\frac{\lambda^{*\alpha}_{ij\beta}}{m}\right].$$

More generally, take $m^\alpha_\gamma = m$ for a single index $\gamma \geqq \beta$ and $m^\sigma_\tau = 0$ otherwise. Then (97.5) gives

$$mb^\gamma_{ik\beta} = \sum_{j=1}^{L_\alpha} \lambda^{*\alpha}_{ij\beta}(b^\alpha_{jk\alpha} + mb^\gamma_{jk\alpha}),$$

and this becomes

$$b^\gamma_{ik\beta} = \sum_{j=1}^{L_\alpha} \lambda^{\alpha\gamma}_{ij\beta} b^\alpha_{jk\alpha} \quad (\gamma \geqq \beta > \alpha), \tag{97.7}$$

when m is allowed to approach zero.

It is clear that $$L_1 \geqq L_2 \geqq L_3 \geqq \ldots \geqq L_n;$$

otherwise a transformation of the independent variables x^α, producing merely a permutation of the indices of these variables, would show that the original choice of the variables x^α was singular, contrary to hypothesis.

Now suppose that equations (97.2) for $\beta = 1$ can be solved for the derivatives

$$\frac{\partial v_1}{\partial x^1}, \frac{\partial v_2}{\partial x^1}, \ldots, \frac{\partial v_{L_1}}{\partial x^1},$$

or in other words that the matrix of the quantities b^1_{jk1}, where $j, k = 1, \ldots, L_1$, is non-singular. Put $\alpha = 1$, $\beta = 2$ in (97.6) and consider the matrix of the quantities λ^1_{ij2} appearing in these equations. If this matrix is of rank R, then $L_2 \leqq R$; this follows from a theorem in Algebra* and the fact that the matrix of the quantities b^2_{ik2} is of rank L_2. Hence $R = L_2$ since R can obviously not be greater than L_2. It then follows from a second theorem in Algebra† and (97.6) that the matrix of the quantities b^2_{ik2} for $i = 1, \ldots, L_2$ and $k = 1, \ldots, L_1$ is of rank L_2. Hence equations (97.2) for $\beta = 2$ can be solved for the derivatives

$$\frac{\partial v_1}{\partial x^2}, \ldots, \frac{\partial v_{L_2}}{\partial x^2}$$

after a suitable choice of the indices of the dependent variables v_k has been made. By a continuation of this process it is evident that, if we change the

* See, for example, Dickson, *Modern Algebraic Theories*, p. 51.

† Dickson, *loc. cit.* p. 51.

notation for the independent variables v_k in accordance with the following scheme:

$$v_{i0} \sim v_{L_1+1}, \ldots, v_w,$$
$$v_{i1} \sim v_{L_2+1}, \ldots, v_{L_1},$$
$$\ldots\ldots\ldots\ldots$$
$$v_{in-1} \sim v_{L_n+1}, \ldots, v_{L_{n-1}},$$
$$v_{in} \sim v_1, \ldots, v_{L_n},$$

equations (97.2) can be written

$$\frac{\partial v_{ik}}{\partial x^\alpha} = \Sigma\,(x, v)\,\frac{\partial v_{pq}}{\partial x^\beta} + \text{✯}, \quad \begin{pmatrix} k = 1, \ldots, n \\ i = 1, \ldots, w_k \\ \alpha = 1, \ldots, k \end{pmatrix} \tag{97.8}$$

where

$$\begin{cases} w_0 = w - L_1, \\ w_1 = L_1 - L_2, \\ w_2 = L_2 - L_3, \\ \ldots\ldots\ldots\ldots \\ w_{n-1} = L_{n-1} - L_n, \\ w_n = L_n, \end{cases} \tag{97.9}$$

and $\beta \geqq \alpha$, $\beta > q$. The coefficients (x, v) in (97.8) depend upon the quantities x^α and v_{rs}; and the ✯ denotes terms containing no derivatives of the v_{rs}; in the sequel the ✯ will be used to denote terms of lower order than those written down explicitly. A system of the form (97.8) will be said to be regular.*

Addition of corresponding members of (97.9) gives

$$w = w_0 + w_1 + \ldots + w_n.$$

98. Extension to tensor differential equations

The results of § 97 can be extended to systems of equations the left members of which are linear in the first derivatives of the components of a tensor T. Thus consider

$$D^{r \ldots s\alpha}_{ip \ldots q}\,\frac{\partial T^{p \ldots q}_{r \ldots s}}{\partial x^\alpha} + \text{✯} = 0 \quad (i = 1, \ldots, L), \tag{98.1}$$

where the coefficients D are functions of the independent variables x^α and the unknowns T; the same is true of the ✯ terms.

* Without the restriction $\beta \geqq \alpha$, the system (97.8) is called regular and immediate by Méray and Riquier, "Sur la convergence des développements des intégrales ordinaires", *Ann. l'éc. norm. sup.* (3), 7 (1890), p. 44.

With respect to a non-singular coordinate system (defined as in § 97) equations (98.1) can be written

$$D^{r\dots s\alpha}_{ip\dots q\beta}\frac{\partial T^{p\dots q}_{r\dots s}}{\partial x^\alpha}+\star=0 \quad \begin{pmatrix}\beta=1,\dots,n\\ i=1,\dots,L_\beta\end{pmatrix}, \tag{98.2}$$

where $D=0$ if $\alpha<\beta$. Under the transformation (97.3) the derivatives of the components of the tensor T transform according to the equations

$$\frac{\partial T^{p\dots q}_{r\dots s}}{\partial x^\alpha}=:\frac{\partial \overline{T}^{a\dots b}_{c\dots d}}{\partial \overline{x}^\sigma}(\delta^\sigma_\alpha+m^\sigma_\alpha)\frac{\partial \overline{x}^c}{\partial x^r}\cdots\frac{\partial \overline{x}^d}{\partial x^s}\frac{\partial x^p}{\partial \overline{x}^a}\cdots\frac{\partial x^q}{\partial \overline{x}^b}.$$

In the coordinate system $\overline{x}$ equations (98.2) therefore take the form

$$(\overline{D}^{c\dots d\alpha}_{ia\dots b\beta}+m^\alpha_\sigma\overline{D}^{c\dots d\sigma}_{ia\dots b\beta})\frac{\partial \overline{T}^{a\dots b}_{c\dots d}}{\partial \overline{x}^\alpha}+\star=0, \tag{98.3}$$

with $\overline{D}^{c\dots d\alpha}_{ia\dots b\beta}=0$ if $\alpha<\beta$. Letting α be a particular number of the set $1,\dots,n-1$ and assuming that $m^\tau_\sigma=0$ if $\alpha\neq\tau$, we obtain by an argument analogous to that employed in § 97 that

$$m^\alpha_\sigma\overline{D}^{c\dots d\sigma}_{ia\dots b\beta}=\sum_{j=1}^{L_\alpha}\lambda^{\star\alpha}_{ij\beta}(\overline{D}^{c\dots d\alpha}_{ja\dots b\alpha}+m^\alpha_\sigma\overline{D}^{c\dots d\sigma}_{ja\dots b\alpha}) \quad (\beta>\alpha,\ \alpha \text{ not summed}) \tag{98.4}$$

as a result of the assumption that the original coordinate system x is non-singular. Then putting $m^\alpha_\gamma=m$ for a single $\gamma\geqq\beta$ and $m^\sigma_\tau=0$ otherwise, we obtain, on allowing m to approach zero, that

$$D^{r\dots s\gamma}_{ip\dots q\beta}=\sum_{j=1}^{L_\alpha}\lambda^{\alpha\gamma}_{ij\beta}D^{r\dots s\alpha}_{jp\dots q\beta} \quad (\gamma\geqq\beta>\alpha). \tag{98.5}$$

If all the components $T^{p\dots q}_{r\dots s}$ are independent, they can be represented by v_k and the system (98.2) together with (98.5) can be written

$$\sum_{k=1}^{w} b^\alpha_{ik\beta}\frac{\partial v_k}{\partial x^\alpha}+c_{i\beta}=0 \quad \begin{pmatrix}\beta=1,\dots,n\\ i=1,\dots,L_\beta\end{pmatrix}, \tag{98.6}$$

$$b^\gamma_{ik\beta}=\sum_{j=1}^{L_\alpha}\lambda^{\alpha\gamma}_{ij\beta}b^\alpha_{jk\alpha} \quad (\gamma\geqq\beta>\alpha), \tag{98.7}$$

where $b^\alpha_{ik\beta}=0$ if $\alpha<\beta$. Suppose, however, that the components $T^{p\dots q}_{r\dots s}$ satisfy linear relations of the form

$$\Sigma T^{k\dots l}_{m\dots n}=0, \tag{98.8}$$

where the indices $k,\dots,l,m,\dots,n$ are obtainable from $p,\dots,q,r,\dots,s$ by permutations. If, in this case, v_k is used to denote the independent components $T^{p\dots q}_{r\dots s}$ when account is taken of (98.8), equations (98.6) and (98.7) will likewise apply. On the basis of the discussion in § 97 the system (98.2) can then be replaced by a system of equations in the regular form (97.8).

99. General existence theorem for regular systems

We shall now impose on the system (97.1) the following two fundamental restrictions.

Condition I. *The coefficients* a^{α}_{ik} *and* c_i *in* (97·1) *are analytic functions in the neighbourhood of some set of values* $x^{\alpha}=q^{\alpha}$ *and* $v_k=(v_k)_q$ *of their arguments.*

Condition II. *The regular system* (97.8) *is completely integrable.*

The first condition carries with it the consequence that the coefficients (x, v) and the ☆ terms in (97.8) are analytic functions in the neighbourhood of some set of values p^{α} and $(v_{ik})_p$, these being values in the neighbourhood of q^{α} and $(v_k)_q$.

By a *principal* derivative we shall mean one which can be obtained by differentiation of a left member of the system under consideration, or one of the derivatives which comprises a left member of the system. All other derivatives will be said to be *parametric*. A system of equations such as (97.8) will be said to be *completely integrable* if the integrability conditions resulting from one differentiation of the system are satisfied identically in the parametric derivatives. It follows that if Conditions I and II are satisfied the system (97.1) has a unique solution, given by a set of convergent power series expansions, corresponding to the arbitrary assignment of analytic data predicted by the form of the left members of the system (97.8) in accordance with the following*

Existence theorem. *Suppose that* (97·1) *is a system such that Conditions I and II are satisfied. Let*

$$\phi_{ik}(x^{k+1}, \ldots, x^n) \quad (i=1, \ldots, w_k),$$

where $k \neq n$, *be an arbitrary function of the variable* $x^{k+1}, \ldots, x^n$, *analytic in the neighbourhood of the values* $x^{\alpha}=p^{\alpha}$ *of their arguments, such that* $\phi_{ik}(p)=(v_{ik})_p$ *for all values of the indices for which the* ϕ_{ik} *are defined. Then there exists one and only one solution* $v_{ik}(x)$ *of* (97.1), *each function* $v_{ik}(x)$ *being analytic in the neighbourhood of the set of values* $x^{\alpha}=p^{\alpha}$, (1) *such that* $v_{in}(p)=(v_{in})_p$ *for* $i=1, \ldots, w_n$, *and* (2) *such that*

$$v_{i0}(x^1, \ldots, x^n)=\phi_{i0}(x^1, \ldots, x^n) \qquad v_{ik}(x^{k+1}, \ldots, x^n)=\phi_{ik}(x^{k+1}, \ldots, x^n).$$
$$(i=1, \ldots, w_0) \qquad \begin{bmatrix} k=1, & \ldots, & n-1 \\ i=1, & \ldots, & w_k \\ x^1=p^1, & \ldots, & x^k=p^k \end{bmatrix}$$

100. Groups of independent components

Equations (54.16) suggest that it is possible to divide the $A(n, 1)$ independent components $A^{\alpha}_{\beta\gamma\delta}$ into $n-1$ groups, namely a group G_0 comprising A_0 components each of which can be taken as a completely arbitrary function

* A direct proof is given by Méray and Riquier, *loc. cit.*

of the n variables x^α, a group G_1 comprising A_1 components each of which can be taken to reduce to an arbitrary function of the $n-1$ variables $x^2, \ldots, x^n$ for $x^1=0$, ..., and a group G_{n-2} comprising A_{n-2} components each of which can be taken to reduce to an arbitrary function of the variables x^{n-1}, x^n for $x^1=\ldots=x^{n-2}=0$. Such a division of the components $A^\alpha_{\beta\gamma\delta}$ into groups G_m must comprise all of the $A(n, 1)$ independent components $A^\alpha_{\beta\gamma\delta}$; this is shown by (54.14). Similar remarks apply to the components $g_{\alpha\beta,\gamma\delta}$ on account of equations (54.15) and (54.17).

Let us first consider the $A^\alpha_{\beta\gamma\delta}$. It is evident that the group G_0 cannot be filled by selecting components $A^\alpha_{\beta\gamma\delta}$ at random from the $A(n, 1)$ independent components $A^\alpha_{\beta\gamma\delta}$, since they are conditioned by equations (46.6). Our problem is to show that it is possible to solve equations (46.6) for first derivatives of the independent components $A^\alpha_{\beta\gamma\delta}$ such that the quantities for which we have solved will fall into $n-2$ mutually exclusive groups; first a group consisting of first derivatives of A_1 of the independent components $A^\alpha_{\beta\gamma\delta}$ with respect to x^1 (these A_1 components will form the group G_1), second a group consisting of first derivatives of A_2 of the independent components $A^\alpha_{\beta\gamma\delta}$ with respect to x^1 and x^2 (these A_2 components will form the group G_2), ..., and finally a group consisting of first derivatives of A_{n-2} of the independent components $A^\alpha_{\beta\gamma\delta}$ with respect to $x^1, x^2, \ldots$ and x^{n-2} (these A_{n-2} components $A^\alpha_{\beta\gamma\delta}$ forming the group G_{n-2}). It is possible to state a rule for the determination of the independent components $A^\alpha_{\beta\gamma\delta}$ in the groups G_m $(m=0, 1, \ldots, n-2)$ as follows(2).

Rule I. The group G_m $(m=0, \ldots, n-2)$ for the components $A^\alpha_{\beta\gamma\delta}$ is composed of all components that can be formed from $A^\alpha_{\beta\gamma\delta}$ by taking $\delta=m+1$; $\alpha=1, \ldots, n$; $\beta, \gamma=1, \ldots, n$ subject to the inequalities $\beta\leqq\gamma$ and $\gamma>m+1$. It can readily be verified that the application of this Rule gives a correct division of the components $A^\alpha_{\beta\gamma\delta}$ into groups G_m. It is to be observed that the components $A^\alpha_{\beta\gamma\delta}$ in a particular group G_m as given by the above Rule are independent among themselves, i.e. are not related by an equation of the form (41.7), and are also independent of the components $A^\alpha_{\beta\gamma\delta}$ in the other groups G_m.

The number of components $A^\alpha_{\beta\gamma\delta}$ in the group G_m $(m=0, \ldots, n-2)$ as determined by the above Rule is $nK(n, 2)-nK(m+1, 2)$ or A_m by (54.18). We shall let A_{lm}, where $l=1, \ldots, A_m$; $m=0, \ldots, n-2$, denote the A_m components $A^\alpha_{\beta\gamma\delta}$ in the group G_m.

The separation of the components $g_{\alpha\beta,\gamma\delta}$ into groups is to be made in an analogous manner.

Rule II. The group G_m $(m=0, \ldots, n-2)$ for the components $g_{\alpha\beta,\gamma\delta}$ is composed of all components that can be formed from $g_{\alpha\beta,\gamma\delta}$ by taking $\alpha=m+1$; $\beta, \gamma, \delta=1, \ldots, n$ subject to the inequalities $\beta\leqq m+1$, $\beta<\gamma$, $\gamma\leqq\delta$ and $\delta>m+1$.

The components $g_{\alpha\beta,\gamma\delta}$ in the above groups G_m are independent since they

are not connected by relations of the form (41.14) or (41.15). The number of components $g_{\alpha\beta,\gamma\delta}$ in the group G_m $(m=0, \ldots, n-2)$ is

$$(m+1)\,K(n,2)-nK(m+1,2),$$

which by (54.19) is equal to B_m. We shall denote by B_{lm}, where $l=1, \ldots, B_m$; $m=0, 1, \ldots, n-2$, the B_m components $g_{\alpha\beta,\gamma\delta}$ in the group G_m.

Finally the enunciation of our existence theorems demands a similar separation of the components of connection $\Gamma^\alpha_{\beta\gamma}$ into groups; we therefore lay down the following

Rule III. The group G_m $(m=0, \ldots, n-1)$ for the components $\Gamma^\alpha_{\beta\gamma}$ is composed of all components that can be formed from $\Gamma^\alpha_{\beta\gamma}$ by taking $\gamma=m+1$; $\alpha, \beta=1, \ldots, n$ subject to the inequality $\beta \leqq m+1$.

There are $\gamma_m=n(m+1)$ components $\Gamma^\alpha_{\beta\gamma}$ in the group G_m and, for convenience of reference, we shall denote these γ_m components by γ_{lm} $(l=1, \ldots, \gamma_m)$. It is obvious that no component γ_{lm} can occur in more than one group, and also that no two components γ_{lm} and γ_{pq} are equal in consequence of the condition (40.2); in fact the components γ_{lm} include all components of affine connection which are not equal in virtue of (40.2).

101. Special case of two dimensions

In the two dimensional space of symmetric affine connection the equations defining the components $B^\alpha_{\beta\gamma\delta}$ of the curvature tensor can be written

$$(101.1)\qquad \begin{cases} \dfrac{\partial\Gamma^\alpha_{12}}{\partial x^1}=\dfrac{\partial\Gamma^\alpha_{11}}{\partial x^2}+B^\alpha_{121}+\Gamma^\alpha_{\sigma 2}\Gamma^\sigma_{11}-\Gamma^\alpha_{\sigma 1}\Gamma^\sigma_{12}, \\[2ex] \dfrac{\partial\Gamma^\alpha_{22}}{\partial x^1}=\dfrac{\partial\Gamma^\alpha_{21}}{\partial x^2}+B^\alpha_{221}+\Gamma^\alpha_{\sigma 2}\Gamma^\sigma_{12}-\Gamma^\alpha_{\sigma 1}\Gamma^\sigma_{22}. \end{cases}$$

Components $B^\alpha_{\beta\gamma\delta}$ other than those which appear in the above equations are linearly dependent on these components. We shall regard the components $B^\alpha_{\beta\gamma\delta}$ in (101.1) as functions of the A_{l0} in accordance with the equations

$$B^\alpha_{\beta\gamma\delta}=:A^\alpha_{\beta\gamma\delta}-A^\alpha_{\beta\delta\gamma}.$$

If we assume the components A_{l0} to be functions of the coordinates x^α, analytic in the neighbourhood of the point $x^\alpha=q^\alpha$, then the Cauchy-Kowalewsky theorem* for the system (101.1) gives us immediately the following (3)

Existence theorem. *Let ϕ_l $(l=1, 2, 3, 4)$ and ψ_p $(p=1, 2)$ denote functions of the variables x^1, x^2, analytic in the neighbourhood of the values $x^1=q^1$, $x^2=q^2$. Also let ζ_q $(q=1, 2, 3, 4)$ denote functions of the variable x^2, which are analytic in the neighbourhood of $x^2=q^2$. Then there exists one, and only one,*

* Systems (101.1) of the type considered by Cauchy and Kowalewsky are a special case of the regular systems for which there are no conditions of integrability.

affine connection with components $\Gamma^{\alpha}_{\beta\gamma}\,(=\Gamma^{\alpha}_{\gamma\beta})$ *in a system of coordinates* x^{α}, *each function* $\Gamma^{\alpha}_{\beta\gamma}(x)$ *being analytic in the neighbourhood of the values* $x^1=q^1$, $x^2=q^2$, *such that*

$$A_{l0}=\phi_l(x^1,x^2) \quad [l=1,2,3,4] \qquad \gamma_{p0}=\psi_p(x^1,x^2) \quad [p=1,2] \qquad \gamma_{q1}=\zeta_q(x^2). \quad \begin{bmatrix} q=1,2,3,4 \\ x^1=q^1 \end{bmatrix}$$

For the metric space we must add to (101.1) those equations which express the conditions that the components $\Gamma^{\alpha}_{\beta\gamma}$ be Christoffel symbols with respect to the components $g_{\alpha\beta}$ of a fundamental metric tensor, i.e.

$$\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}=g_{\sigma\beta}\Gamma^{\sigma}_{\alpha\gamma}+g_{\alpha\sigma}\Gamma^{\sigma}_{\beta\gamma}. \tag{101.2}$$

We now consider that the components $B^{\alpha}_{\beta\gamma\delta}$ in (101.1) are functions of the $g_{\alpha\beta}$ and $g_{\alpha\beta,\gamma\delta}$, namely

$$B^{\alpha}_{\beta\gamma\delta}=g^{\alpha\sigma}(g_{\sigma\gamma,\beta\delta}-g_{\sigma\delta,\beta\gamma})$$

[see equations (49.9)]. Now differentiate (101.2) with respect to x^{δ} and eliminate the second derivatives of $g_{\alpha\beta}$ by the condition

$$\frac{\partial^2 g_{\alpha\beta}}{\partial x^{\gamma}\partial x^{\delta}}=\frac{\partial^2 g_{\alpha\beta}}{\partial x^{\delta}\partial x^{\gamma}}.$$

From the resulting equations eliminate the first derivatives of $\Gamma^{\alpha}_{\beta\gamma}$ and $g_{\alpha\beta}$ by means of equations (101.1) and (101.2) themselves. We thus obtain a system of equations which is readily shown to be satisfied identically. Taking the single independent component B_{10} and the γ_{l0} as known functions of the variables x^{α}, the system composed of (101.1) and (101.2) is therefore completely integrable.

EXISTENCE THEOREM. *Let* ϕ *and* ψ_p $(p=1,2)$ *denote functions of the variables* x^1, x^2, *analytic in the neighbourhood of the values* $x^1=q^1$, $x^2=q^2$. *Let* ζ_q $(q=1,2,3,4)$ *denote functions of the variable* x^2, *which are analytic in the neighbourhood of* $x^2=q^2$. *And finally let* $(g_{\alpha\beta})_q=(g_{\beta\alpha})_q$ *denote arbitrary constants such that the determinant* $|(g_{\alpha\beta})_q|\neq 0$. *Then there exists one, and only one, fundamental metric tensor with components* $g_{\alpha\beta}\,(=g_{\beta\alpha})$ *in a system of coordinates* x^{α}, *each function* $g_{\alpha\beta}(x)$ *being analytic in the neighbourhood of the values* $x^1=q^1$, $x^2=q^2$, *such that* $g_{\alpha\beta}(q)=(g_{\alpha\beta})_q$ *and*

$$B_{10}=\phi(x^1,x^2) \qquad \gamma_{p0}=\psi_p(x^1,x^2) \quad [p=1,2] \qquad \gamma_{q1}=\zeta_q(x^2). \quad \begin{bmatrix} q=1,2,3,4 \\ x^1=q^1 \end{bmatrix}$$

102. GENERAL CASE OF n ($\geqq 3$) DIMENSIONS

When the dimensionality n is greater than or equal to three, the systems (46.6) and (46.12) must be considered; in the two dimensional case treated

in § 101 these systems were satisfied identically (see § 54). Taking first the equations (46.6) we shall show that these can be put into the form

$$(102.1) \qquad \frac{\partial A_{ik}}{\partial x^{\alpha}} = \Sigma \frac{\partial A_{pq}}{\partial x^{r}} \cdot \quad \wedge \qquad \left(\begin{array}{l} k = 1, \ldots, n-2 \\ i = 1, \ldots, A_k \\ \alpha = 1, \ldots, k \\ r \geqq \alpha, \; r > q \end{array} \right)$$

in which the summation Σ denotes a linear expression in the derivatives $\partial A^{\alpha}_{\beta\gamma\delta}/\partial x^{\epsilon}$ with constant coefficients; the ☆ terms are bilinear forms in the components $A^{\alpha}_{\beta\gamma\delta}$ and $\Gamma^{\alpha}_{\beta\gamma}$. For this purpose we write the system (46.6) in the form

$$(102.2) \qquad \frac{\partial A^{\alpha}_{\beta\gamma\delta}}{\partial x^{\epsilon}} = \frac{\partial A^{\alpha}_{\beta\gamma\epsilon}}{\partial x^{\delta}} + \frac{2}{3}\frac{\partial A^{\alpha}_{\beta\epsilon\delta}}{\partial x^{\gamma}} + \frac{1}{3}\frac{\partial A^{\alpha}_{\delta\epsilon\beta}}{\partial x^{\gamma}} + \frac{1}{3}\frac{\partial A^{\alpha}_{\epsilon\gamma\delta}}{\partial x^{\beta}} \cdot \quad \frac{1}{3}\frac{\partial A^{\alpha}_{\delta\gamma\epsilon}}{\partial x^{\beta}}$$

and consider the conditions on the indices $\delta = \mu + 1$, $\gamma > \delta > \epsilon$, $\beta \leqq \gamma$ to be satisfied, where $\mu = 1, \ldots, n-2$. The component $A^{\alpha}_{\beta\gamma\delta}$ whose derivative forms the left member of (102.2) therefore belongs to the group G_{μ}. Let us now examine the components A whose derivatives are in the right members of (102.2) bearing in mind that $\delta = \mu + 1$ and that β, γ, δ, ϵ satisfy the above inequalities. The first component $A^{\alpha}_{\beta\gamma\epsilon}$ belongs to group $G_{\epsilon-1}$. The component $A^{\alpha}_{\beta\epsilon\delta}$ belongs to group G_{μ} if $\beta > \mu + 1$. If $\beta = \mu + 1$, it belongs to group $G_{\epsilon-1}$. Lastly if $\beta < \mu + 1$ the component $A^{\alpha}_{\beta\epsilon\delta}$ is to be put equal to $-(A^{\alpha}_{\epsilon\delta\beta} + A^{\alpha}_{\delta\beta\epsilon})$, of which $A^{\alpha}_{\epsilon\delta\beta}$ belongs to group $G_{\beta-1}$, and $A^{\alpha}_{\delta\beta\epsilon}$ belongs to group $G_{\epsilon-1}$. The component $A^{\alpha}_{\delta\epsilon\beta}$ belongs to group $G_{\beta-1}$ if $\beta \leqq \mu$ and to group $G_{\epsilon-1}$ if $\beta = \mu + 1$. If $\beta > \mu + 1$ the component $A^{\alpha}_{\delta\epsilon\beta}$ is to be put equal to $-(A^{\alpha}_{\epsilon\beta\delta} + A^{\alpha}_{\beta\delta\epsilon})$, where $A^{\alpha}_{\epsilon\beta\delta}$ belongs to group G_{μ}, and $A^{\alpha}_{\beta\delta\epsilon}$ to group $G_{\epsilon-1}$. We see therefore that the derivatives in the first three terms in the right members of (102.2) yield derivatives of the type occurring in the right members of (102.1).

Since $\gamma \geqq \mu + 2$ the component $A^{\alpha}_{\epsilon\gamma\delta}$ belongs to group G_{μ}. If $\beta \geqq \mu + 1$, we have $\epsilon < \beta$ so that the fourth term in the right members of (102.2) will be of the required sort. However, when $\beta \leqq \mu$ this term will contribute a derivative of the type of those in the left members of (102.1). We then eliminate it by means of (102.2), i.e. by a substitution of the form

$$\frac{\partial A^{\alpha}_{\epsilon\gamma\delta}}{\partial x^{\beta}} = \ldots.$$

In case $\epsilon > \beta - 1$, this substitution will yield derivatives of the type occurring in the right members of (102.1) in their relation to the derivatives in the left members of these equations;* otherwise a second substitution of the above sort is required. An analogous treatment of the derivative of the component $A^{\alpha}_{\delta\gamma\epsilon}$ which belongs to the group $G_{\epsilon-1}$ results finally in the derivation of the equations (102.1).

* It is to be noted that one term will give the same derivative as that occurring on the left side of the equation, but with a coefficient different from one.

As a result of the conditions (102.1) the number of algebraically independent components $A^{\alpha}_{\beta\gamma\delta,\epsilon}$ is

$$n\left(\sum_{\alpha=0}^{n-2} A_{\alpha}\right)-\sum_{\alpha=1}^{n-2}\alpha A_{\alpha},$$

which is seen to be equal to $A(n,2)$ as defined in §54. Equations (46.6) cannot therefore furnish additional conditions on the components over those which result from (102.1); in other words the equations (46.6) and (102.1) are equivalent algebraically.

In place of (101.1) we now have, as the equations which define the components $B^{\alpha}_{\beta\gamma\delta}$ of the curvature tensor, the equations

$$\frac{\partial\Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}}=\frac{\partial\Gamma^{\alpha}_{\beta\delta}}{\partial x^{\gamma}}+B^{\alpha}_{\beta\gamma\delta}+\Gamma^{\alpha}_{\sigma\gamma}\Gamma^{\sigma}_{\beta\delta}-\Gamma^{\alpha}_{\sigma\delta}\Gamma^{\sigma}_{\beta\gamma}\quad\begin{pmatrix}\gamma=2,\ldots,n\\ \beta=1,\ldots,\gamma\\ \delta=1,\ldots,\gamma-1\end{pmatrix}.$$

Let us observe that, in terms of the independent components γ_{lm} and A_{lm}, these equations have the form

$$\frac{\partial\gamma_{ik}}{\partial x^{\alpha}}=\frac{\partial\gamma_{pq}}{\partial x^{r}}+\Sigma A_{pq}+\Sigma\gamma_{pq}\gamma_{uv}\quad\begin{pmatrix}k=1,\ldots,n-1\\ i=1,\ldots,\gamma_k\\ \alpha=1,\ldots,k\\ r\geqq\alpha,\ r>q\end{pmatrix}\tag{102.3}$$

where the first summation Σ denotes a linear form, and the second denotes a quadratic form, in the variables indicated.

The system composed of (102.1) and (102.3) is regular. Moreover this system is completely integrable. This can be seen from the fact that if we form the conditions of integrability we obtain a system R involving the following quantities

$$\left.\begin{aligned}&\gamma_{pq};\ \frac{\partial\gamma_{pq}}{\partial x^{r}};\ \frac{\partial^2\gamma_{pq}}{\partial x^{r}\partial x^{s}};\\ &\begin{pmatrix}q=0,\ldots,n-1;\ p=1,\ldots,\gamma_q\\ r,s=1,\ldots,n;\ r\geqq s;\ r,s>q\end{pmatrix}\\ &A_{pq};\ \frac{\partial A_{pq}}{\partial x^{r}};\ \frac{\partial^2 A_{pq}}{\partial x^{r}\partial x^{s}};\\ &\begin{pmatrix}q=0,\ldots,n-2;\ p=1,\ldots,A_q\\ r,s=1,\ldots,n;\ r\geqq s;\ r,s>q\end{pmatrix}\end{aligned}\right.\tag{102.4}$$

Now at a point Q of the space there are

$$\left\{\begin{array}{llll} nK(n,2) & \text{independent} & \text{components} & \Gamma^{\alpha}_{\beta\gamma}\\ nK(n,3) & ,, & ,, & \Gamma^{\alpha}_{\beta\gamma\delta},\\ nK(n,4) & ,, & ,, & \Gamma^{\alpha}_{\beta\gamma\delta\epsilon},\\ A(n,1) & ,, & ,, & A^{\alpha}_{\beta\gamma\delta},\\ A(n,2) & ,, & ,, & A^{\alpha}_{\beta\gamma\delta\epsilon},\\ A(n,3) & ,, & ,, & A^{\alpha}_{\beta\gamma\delta\epsilon\eta}\end{array}\right.\tag{102.5}$$

(see § 38 and § 54). If we denote the sum of the components Γ and A in (102.5) by S, so that

$$S = nK(n,2) + nK(n,3) + nK(n,4) + A(n,1) + A(n,2) + A(n,3),$$

then S also gives the number of quantities in (102.4); in fact the number of quantities in any one of the six sets in (102.4) is equal to the corresponding term in the sum S. When the quantities in (102.4) are given at a point Q of the space, the components Γ and A in (102.5) are determined at Q. It follows that all of the quantities in (102.4) can be given arbitrary values at the point Q, for, if this were not the case, the number of independent components Γ and A in (102.5) would be less than S. Hence the equations of the system R must be satisfied identically, and we have the following

EXISTENCE THEOREM. *Let*

$$\phi_{lm}(x^{m+1}, \ldots, x^n) \quad \begin{bmatrix} m = 0, \ldots, n-2 \\ l = 1, \ldots, A_m \end{bmatrix} \qquad \psi_{lm}(x^{m+1}, \ldots, x^n) \quad \begin{bmatrix} m = 0, \ldots, n-1 \\ l = 1, \ldots, \gamma_m \end{bmatrix}$$

denote arbitrary functions of the variables $x^{m+1}, \ldots, x^n$, analytic in the neighbourhood of the values $x^\alpha = q^\alpha$ of their arguments. Then there exists one, and only one, affine connection with components $\Gamma^\alpha_{\beta\gamma}\ (= \Gamma^\alpha_{\gamma\beta})$ in a system of coordinates x^α, each function $\Gamma^\alpha_{\beta\gamma}(x)$ being analytic in the neighbourhood of the values $x^\alpha = q^\alpha$, such that

$$A_{l0} = \phi_{l0}(x^1, \ldots, x^n) \quad [l = 1, \ldots, A_0] \qquad \gamma_{l0} = \psi_{l0}(x^1, \ldots, x^n) \quad [l = 1, \ldots, \gamma_0]$$

$$A_{lm} = \phi_{lm}(x^{m+1}, \ldots, x^n) \quad \begin{bmatrix} m = 1, \ldots, n-2 \\ l = 1, \ldots, A_m \\ x^1 = q^1, \ldots, x^m = q^m \end{bmatrix} \qquad \gamma_{lm} = \psi_{lm}(x^{m+1}, \ldots, x^n) \quad \begin{bmatrix} m = 1, \ldots, n-1 \\ l = 1, \ldots, \gamma_m \\ x^1 = q^1, \ldots, x^m = q^m \end{bmatrix}$$

Turning, now, to the metric case we can derive the system

$$\frac{\partial B_{ik}}{\partial x^\alpha} = \Sigma \frac{\partial B_{pq}}{\partial x^r} + \star \quad \begin{pmatrix} k = 1, \ldots, n-2 \\ i = 1, \ldots, B_k \\ \alpha = 1, \ldots, k \\ r \geqq \alpha,\ r > q \end{pmatrix} \tag{102.6}$$

from (46.12) by a consideration analogous to that employed in the derivation of (102.1); the summation Σ in (102.6) denotes a linear form in the derivatives $\partial g_{\alpha\beta,\gamma\delta}/\partial x^\epsilon$ with constant coefficients and the $\star$ terms represent bilinear forms in the components $g_{\alpha\beta,\gamma\delta}$ and $\Gamma^\alpha_{\beta\gamma}$. Then the system composed of (102.6), (102.3) and (101.2) is regular and can be shown to be completely integrable by an argument analogous to that used in the affine case.

EXISTENCE THEOREM. *Let*

$$\phi_{lm}(x^{m+1}, \ldots, x^n) \quad \begin{bmatrix} m = 0, \ldots, n-2 \\ l = 1, \ldots, B_m \end{bmatrix} \qquad \psi_{lm}(x^{m+1}, \ldots, x^n) \quad \begin{bmatrix} m = 0, \ldots, n-1 \\ l = 1, \ldots, \gamma_m \end{bmatrix}$$

denote arbitrary functions of the variables $x^{m+1}, \ldots, x^n$, *analytic in the neighbourhood of the values* $x^\alpha = q^\alpha$ *of their arguments. Also let* $(g_{\alpha\beta})_q = (g_{\beta\alpha})_q$ *denote arbitrary constants, such that the determinant* $|(g_{\alpha\beta})_q| \neq 0$. *Then there exists one, and only one, fundamental metric tensor with components* $g_{\alpha\beta}(=g_{\beta\alpha})$ *in a system of coordinates* x^α, *each function* $g_{\alpha\beta}(x)$ *being analytic in the neighbourhood of the values* $x^\alpha = q^\alpha$, *such that* $g_{\alpha\beta}(q) = (g_{\alpha\beta})_q$ *and*

$$B_{l0} = \phi_{l0}(x^1, \ldots, x^n) \quad [l = 1, \ldots, B_0] \qquad \gamma_{l0} = \psi_{l0}(x^1, \ldots, x^n) \quad [l = 1, \ldots, \gamma_0]$$

$$B_{lm} = \phi_{lm}(x^{m+1}, \ldots, x^n) \quad \begin{bmatrix} m = 1, \ldots, n-2 \\ l = 1, \ldots, B_m \\ x^1 = q^1, \ldots, x^m = q^m \end{bmatrix} \qquad \gamma_{lm} = \psi_{lm}(x^{m+1}, \ldots, x^n) \quad \begin{bmatrix} m = 1, \ldots, n-1 \\ l = 1, \ldots, \gamma_m \\ x^1 = q^1, \ldots, x^m = q^m \end{bmatrix}$$

103. The existence theorems in normal coordinates

The special case $n = 2$ has already been treated in § 54; we proceed therefore to the general case $n \geqq 3$.

Let us write the equations (102.1) in terms of a system of normal coordinates y^α as

$$\text{(103.1)} \qquad \frac{\partial A_{lm}}{\partial y^\alpha} = \Sigma \frac{\partial A_{pq}}{\partial y^r} + ☆ \qquad \begin{pmatrix} m = 1, \ldots, n-2 \\ l = 1, \ldots, A_m \\ \alpha = 1, \ldots, m \\ \alpha \leqq r, \; q < r \end{pmatrix}$$

where now the ☆ represents a bilinear form in the variables $C^\alpha_{\beta\gamma}$ and A_{uv}. We wish to show that the equations obtained by differentiation of (103.1) with respect to $y^{\beta_1}, \ldots, y^{\beta_s}$ can be put in the form

$$\text{(103.2)} \qquad \frac{\partial^{s+1} A_{lm}}{\partial y^\alpha \partial y^{\beta_1} \ldots \partial y^{\beta_s}} = \Sigma \frac{\partial^{s+1} A_{pq}}{\partial y^r \partial y^{\sigma_1} \ldots \partial y^{\sigma_s}} + \Sigma \frac{\partial^s ☆}{\partial y^{\beta_1} \ldots \partial y^{\beta_s}},$$

where $\qquad s = 1, 2, \ldots; \quad \beta_1, \ldots, \beta_s = 1, 2, \ldots, n;$

and the inequalities $r, \sigma_1, \ldots, \sigma_s > q$ are satisfied. Each of the summations represents a linear homogeneous expression with constant coefficients in the derivatives of the type indicated. It is evident that (103.2) holds for

$$\beta_1, \ldots, \beta_s = n-1, n,$$

since (103.2) then results directly by differentiation of (103.1). Now differentiate any equation (103.2) for

$$\beta_2, \ldots, \beta_s = n-1, n$$

with respect to y^{n-2}. Then all derivatives in the right member of the resulting equation will be of the type which appears in the right member of (103.2) except derivatives of A_{pn-2}. These latter derivatives, however, can be eliminated by a substitution (103.2) for

$$\beta_1, \ldots, \beta_s = n-1, n$$

since $r > q$; hence (103.2) is true for

$$\beta_1 = n-2;\quad \beta_2, \ldots, \beta_s = n-1, n.$$

Differentiating any equation (103.2) for

$$\beta_1 = n-2;\quad \beta_3, \ldots, \beta_s = n-1, n$$

with respect to y^{n-2}, we find in a similar manner that (103.2) is true for

$$\beta_1, \beta_2 = n-2;\quad \beta_3, \ldots, \beta_s = n-1, n;\quad \text{etc.}$$

Hence (103.2) holds for

$$\beta_1, \ldots, \beta_s = n-2, n-1, n.$$

If we now differentiate any equation (103.2) for

$$\beta_2, \ldots, \beta_s = n-2, n-1, n$$

with respect to y^{n-3} and eliminate the derivatives of A_{pn-3}, A_{pn-2} which occur, we obtain (103.2) for

$$\beta_1 = n-3;\quad \beta_2, \ldots, \beta_s = n-2, n-1, n.$$

Then differentiating (103.2) for

$$\beta_1 = n-3;\quad \beta_3, \ldots, \beta_s = n-2, n-1, n,$$

we obtain (103.2) for

$$\beta_1, \beta_2 = n-3;\quad \beta_3, \ldots, \beta_s = n-2, n-1, n, \text{etc.}$$

Hence (103.2) is true for

$$\beta_1, \ldots, \beta_s = n-3, n-2, n-1, n.$$

Continuing this process we finally obtain (103.2) for

$$\beta_1, \ldots, \beta_s = 1, \ldots, n,$$

as was to be proved.

Let us denote by ϕ_{lm}, where

$$l = 1, \ldots, A_m;\quad m = 0, 1, \ldots, n-2,$$

an arbitrary function of the variables $y^{m+1}, y^{m+2}, \ldots, y^n$, analytic in the neighbourhood of the values $y^{m+1} = \ldots = y^n = 0$. Now put

$$A_{l0} = \phi_{l0}\quad (l = 1, \ldots, A_0);$$

also put $$A_{lm} = \phi_{lm}\quad (l = 1, \ldots, A_m;\ m = 1, \ldots, n-2)$$

for $y^1 = \ldots = y^m = 0$. From the functions ϕ_{lm} and equations (103.2) we can then calculate the successive coefficients $A^{\alpha}_{\beta\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)$ in the power series expansions of the functions $A^{\alpha}_{\beta\gamma\delta}$ about the point $y^{\alpha} = 0$, i.e. we can determine the series

$$\text{(103.3)}\qquad A^{\alpha}_{\beta\gamma\delta} = A^{\alpha}_{\beta\gamma\delta}(0) + A^{\alpha}_{\beta\gamma\delta,\epsilon_1}(0)\, y^{\epsilon_1} + \frac{1}{2!} A^{\alpha}_{\beta\gamma\delta,\epsilon_1\epsilon_2}(0)\, y^{\epsilon_1} y^{\epsilon_2} + \ldots,$$

where the constant terms $A^{\alpha}_{\beta\gamma\delta}(0)$ are given by the functions ϕ_{lm} evaluated at $y^{\alpha} = 0$, use being made of the identities (41.7). In the determination of the quantities $A^{\alpha}_{\beta\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)$ the initial conditions $C^{\alpha}_{\beta\gamma} = 0$ at $y^{\alpha} = 0$ are to be imposed so that the derivatives of the ☆ terms occurring in (103.2) involve only derivatives of the A_{uv} which are of lower order than those occurring on the left side of (103.2); also $\partial C^{\alpha}_{\beta\gamma}/\partial y^{\delta}$ at $y^{\alpha} = 0$ is equal to $A^{\alpha}_{\beta\gamma\delta}(0)$ and the

higher derivatives of $C^{\alpha}_{\beta\gamma}$ at $y^{\alpha}=0$ are determined by equations of the type (43.9). The number of arbitrary quantities $A^{\alpha}_{\beta\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)$ is

$$A_0 K(n,r)+A_1 K(n-1,r)+\ldots+A_{n-2} K(2,r),$$

which is equal to $A(n,1+r)$ by (54.16). These $A(n,1+r)$ arbitrary quantities $A^{\alpha}_{\beta\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)$ being given, the remainder are determined by the above calculation and by the identities obtainable from (41.7) by extension. The determination of the quantities $A^{\alpha}_{\beta\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)$ will be unique, for if this were not the case we would be led to equations as a result of conditions of integrability which would reduce the number of arbitrary quantities $A^{\alpha}_{\beta\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)$ to a number less than $A(n,1+r)$. Equations (46.5) cannot therefore give additional conditions on the quantities $A^{\alpha}_{\beta\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)$ so that the series (103.3) if convergent will define a set of functions $A^{\alpha}_{\beta\gamma\delta}(y)$ which will be the components of a normal tensor in a system of normal coordinates. Since it will be shown in § 104 that the formal power series (103.3) converges, it follows that by Theorem A, § 46 we therefore have the following

EXISTENCE THEOREM. *Let $\phi_{lm}(y)$, where $l=1,\ldots,A_m$; $m=0,1,\ldots,n-2$, denote an arbitrary function of the variables $y^{m+1}, y^{m+2},\ldots,y^n$ which is analytic in the neighbourhood of the values $y^{m+1}=\ldots=y^n=0$; also let A_{lm}, where $l=1,\ldots,A_m$, denote the components $A^{\alpha}_{\beta\gamma\delta}$ in the group G_m $(m=0,1,\ldots,n-2)$. Then there exists one, and only one, affine connection with components $C^{\alpha}_{\beta\gamma}(y)=C^{\alpha}_{\gamma\beta}(y)$ in a system of normal coordinates y^{α}, each function $C^{\alpha}_{\beta\gamma}$ being analytic in the neighbourhood of the values $y^{\alpha}=0$, such that the independent components A_{lm} of the resulting normal tensor are*

$$A_{l0}=\phi_{l0}\quad (l=1,\ldots,A_0),$$

and

$$A_{lm}=\phi_{lm}\quad (l=1,\ldots,A_m;\ m=1,\ldots,n-2)$$

for $y^1=\ldots=y^m=0$.

In the metric case we proceed in an analogous manner. If we write the equations (102.6) in terms of a system of normal coordinates y^{α}, we can derive equations of the form (103.2) for the independent components B_{lm} of the metric normal tensor. Then denoting by θ_{lm}, where

$$l=1,\ldots,B_m;\quad m=0,1,\ldots,n-2,$$

an arbitrary function of the variables $y^{m+1},\ldots,y^n$, analytic in the neighbourhood of the values $y^{m+1}=\ldots=y^n=0$, we put

$$B_{l0}=\theta_{l0}\quad (l=1,\ldots,B_0),$$

and for $y^1=\ldots=y^m=0$ we put

$$B_{lm}=\theta_{lm}\quad (l=1,\ldots,B_m;\ m=1,\ldots,n-2).$$

Now assign the initial conditions $\psi_{\alpha\beta}=(g_{\alpha\beta})_0=(g_{\beta\alpha})_0$, such that the determinant $|(g_{\alpha\beta})_0|\neq 0$, and $\partial\psi_{\alpha\beta}/\partial y^{\gamma}=0$ at $y^{\alpha}=0$; also put $\partial^2\psi_{\alpha\beta}/\partial y^{\gamma}\partial y^{\delta}$ at $y^{\alpha}=0$ equal to $g_{\alpha\beta,\gamma\delta}(0)$, which is determined from the given functions θ_{lm}, and determine the higher derivatives of $\psi_{\alpha\beta}$ at $y^{\alpha}=0$ by equations of the type (43.15). Then as in the affine case we can determine uniquely, use being

made of (41.14) and (41.15), the successive coefficients $g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)$ in the series

$$(103.4)\qquad g_{\alpha\beta,\gamma\delta} = g_{\alpha\beta,\gamma\delta}(0) + g_{\alpha\beta,\gamma\delta,\epsilon_1}(0)\, y^{\epsilon_1} + \frac{1}{2!} g_{\alpha\beta,\gamma\delta,\epsilon_1\epsilon_2}(0)\, y^{\epsilon_1} y^{\epsilon_2} + \ldots.$$

The number of arbitrary quantities $g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)$ is $G(n, 2+r)$ by (54.17), and since the remaining quantities $g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)$ are determined by the above procedure when these arbitrary ones are given, the equations (46.10) cannot furnish additional conditions on the quantities $g_{\alpha\beta,\gamma\delta,\epsilon_1\ldots\epsilon_r}(0)$. It will be shown in § 105 that the series (103.4) converges; hence by Theorem B, § 46, we have the following

Existence theorem. *Let θ_{lm}, where $l=1, \ldots, B_m$; $m=0, 1, \ldots, n-2$, denote an arbitrary function of the variables $y^{m+1}, y^{m+2}, \ldots, y^n$ which is analytic in the neighbourhood of the values $y^{m+1} = \ldots = y^n = 0$; let $(g_{\alpha\beta})_0 = (g_{\beta\alpha})_0$, where $\alpha, \beta = 1, \ldots, n$, be arbitrary constants such that the determinant $|(g_{\alpha\beta})_0| \neq 0$; also let B_{lm}, where $l=1, \ldots, B_m$, denote the components $g_{\alpha\beta,\gamma\delta}$ in the group G_m $(m=0, 1, \ldots, n-2)$. Then there exists one, and only one, fundamental tensor with components $\psi_{\alpha\beta}(y) = \psi_{\beta\alpha}(y)$ in a system of normal coordinates y^α, each function $\psi_{\alpha\beta}$ being analytic in the neighbourhood of the values $y^\alpha = 0$, such that $\psi_{\alpha\beta}(0) = (g_{\alpha\beta})_0$, and such that the independent components B_{lm} of the resulting metric normal tensor are*

$$B_{l0} = \theta_{l0} \quad (l = 1, \ldots, B_0),$$

and
$$B_{lm} = \theta_{lm} \quad (l=1, \ldots, B_m;\ \ m = 1, \ldots, n-2)$$

for $y^1 = \ldots = y^m = 0$.

104. Convergence of the A series

In order to prove the convergence of the series (103.3) we must treat the combined system of equations (103.1) and (46.1). For this purpose let us consider a system of differential equations(4)

$$(104.1a)\qquad \frac{b_m}{a_\alpha}\frac{\partial \mathfrak{A}_{lm}}{\partial y^\alpha} = DF(y) + \frac{\mu\Theta(\mathfrak{A})}{1-\epsilon}\,\Sigma\mathfrak{C}^\epsilon_{\beta\gamma},$$

$$(104.1b)\qquad \frac{1}{a_\delta}\frac{\partial \mathfrak{C}^\alpha_{\beta\gamma}}{\partial y^\delta} = K\Sigma b_v \mathfrak{A}_{uv} + \frac{4}{R}\,\Sigma\mathfrak{C}^\lambda_{\mu\nu}\mathfrak{C}^\rho_{\sigma\tau},$$

in which the a_α, b_m and μ are positive constants greater than unity whose more exact values will be fixed later. The positive constants D, K, and R are to be taken such that

$$(104.2)\qquad D \geqq \frac{\nu_m}{a_\alpha};\ \ K \geqq \frac{}{aR},\ \ R \leqq \frac{a_\beta}{a_\gamma},\ \ R<1;\ \ 1<a \leqq a_\gamma,$$

where a is equal to the least of the constants a_α. The indices l, m and u, v in (104.1) assume all possible values for the corresponding indices on the components A_{uv}; other indices in (104.1) assume values from 1 to n inclusive. All summations in (104.1) denote the sum of all terms that can be formed from the representative term by giving all possible values to the indices that occur. We will also use sets of positive constants $\nu_1, \ldots, \nu_m$ such that the sum $\nu_1 + \ldots + \nu_m$ is equal to the positive quantity $\epsilon\,(<1)$ occurring in (104.1a). The integer m in the sum $\nu_1 + \ldots + \nu_m$ in any particular equation (104.1a) which corresponds to an equation (103.1) is equal to the number of different derivatives $\partial A_{pq}/\partial y^r$ in the corresponding equation (103.1), i.e. the equation (103.1) for which the indices l, m, α have the same values as in the equation (104.1a) under consideration. The integer m for an equation (104.1a) which does not correspond to an

equation (103.1) can be taken to have the value unity. The function $F(y)$ in (104.1a) is defined by

$$F(y)=\frac{\Omega}{1-(\Sigma a_\beta y^\beta)/\rho}, \tag{104.3}$$

where the positive constants Ω, ρ are to be chosen so that the expression

$$\frac{\Omega}{1-(a_{m+1}y^{m+1}+\ldots+a_n y^n)/\rho} \tag{104.4}$$

is dominant for each derivative $\partial\phi_{lm}/\partial y^\alpha$ $(\alpha>m)$ of the A_m functions ϕ_{lm} introduced in § 103. Finally

$$\Theta(\mathfrak{A})=\{1-\Sigma b_v[\mathfrak{A}_{uv}-(\mathfrak{A}_{uv})_0]\}^{-1},$$

where Σ denotes a summation on the indices u, v over all their possible values.

Equations (104.1) constitute a completely integrable system of total differential equations. Hence, according to the well-known theorem for the existence of solutions of systems of total differential equations, there exists a unique solution $\mathfrak{A}_{lm}(y)$, $\mathfrak{C}^\alpha_{\beta\gamma}(y)$ of the system (104.1) such that these integrals assume an arbitrary set of initial values $(\mathfrak{A}_{lm})_0$ and $(\mathfrak{C}^\alpha_{\beta\gamma})_0$. We shall choose the initial values $(\mathfrak{C}^\alpha_{\beta\gamma})_0$ or $\mathfrak{C}^\alpha_{\beta\gamma}(0)$ of the functions $\mathfrak{C}^\alpha_{\beta\gamma}(y)$ so that (1) $\mathfrak{C}^\alpha_{\beta\gamma}(0)=\mathfrak{C}^\alpha_{\gamma\beta}(0)$ and (2) so that the inequalities

$$\mathfrak{C}^\alpha_{\beta\gamma}\geqq 0 \tag{104.5}$$

are satisfied; the initial values $(\mathfrak{A}_{lm})_0$ or $\mathfrak{A}_{lm}(0)$ of the functions $\mathfrak{A}_{lm}(y)$ will be chosen so that

$$\mathfrak{A}_{lm}(0)\geqq|(A_{lm})_0|. \tag{104.6}$$

It is to be observed that the integrals $\mathfrak{C}^\alpha_{\beta\gamma}(y)$ are symmetric in their lower indices. Furthermore all derivatives at $y^\alpha=0$ of the integrals $\mathfrak{A}_{lm}$, $\mathfrak{C}^\alpha_{\beta\gamma}$ are essentially positive, in fact any derivative of one of the sets of terms in the right members of (104.1) or any derivative of the function $\Theta(\mathfrak{A})$ is expressible as a polynomial in positive or zero elements—this polynomial being constructed from its elements entirely by the operations of addition and multiplication. Hence we have that

$$\mathfrak{A}_{lm\,|\,\alpha_1\ldots\alpha_s}(0)\geqq\left|\left(\frac{\partial^s\phi}{\partial y^{\alpha_1}\ldots\partial y^{\alpha_s}}\right)_0\right| \tag{104.7}$$

where $\alpha_1,\ldots,\alpha_s>m$, and $\mathfrak{A}_{lm\,|\,\alpha_1\ldots\alpha_s}$ denotes the derivative of the function $\mathfrak{A}_{lm}$. This inequality follows obviously from the inequalities (104.2) and the dominant property of the function $F(y)$ in (104.1a).

Now the equations $\dfrac{b_m}{a_\alpha}\dfrac{\partial\mathfrak{A}_{lm}}{\partial y^\alpha}=\dfrac{b_q}{a_r}\dfrac{\partial\mathfrak{A}_{pq}}{\partial y^r}$

follow from (104.1a) for all values of the indices involved. It is therefore possible to write those equations (104.1a) which correspond to equations (103.1) in the form

$$\frac{\partial\mathfrak{A}_{lm}}{\partial y^\alpha}=\Sigma\nu\frac{a_\alpha}{a_r}\frac{b_q}{b_m}\frac{\partial\mathfrak{A}_{pq}}{\partial y^r}+\frac{a_\alpha}{b_m}D(1-\epsilon)F(y)+\frac{a_\alpha}{b_m}\mu\Theta(\mathfrak{A})\Sigma\mathfrak{C}^\epsilon_{\beta\gamma}, \tag{104.8}$$

where the quantity ν assumes the values ν_1 to ν_m so that the first summation in these equations denotes a sum of m terms in which the derivatives are taken to correspond to derivatives in the summation in the right members of the corresponding equations (103.1). The coefficients in equations (103.1) are dominated by the corresponding coefficients in the above equations (104.8) in consequence of the following

LEMMA. *Given any two positive constants P and Q, it is possible to assign values, each of which is greater than P, to the constants $a_1,\ldots,a_n$ and $b_0,\ldots,b_n$, such that each of the coefficients*

$$\nu\frac{a_\alpha}{a_r}\frac{b_q}{b_m} \tag{104.9}$$

of the derivatives in the right members of (104.8) *will be greater than Q.*

To prove this lemma we take $a_\alpha=\rho\gamma^{-\alpha}$ and $b_m=\sigma\gamma^{-m}$, where ρ, σ are constants and γ is a constant satisfying the inequalities $\gamma>1$, $\gamma>Q/\delta$, in which δ is the least of the ν's. The above coefficient then becomes

$$\nu\gamma^{[(m-\alpha)-(q-r)]}.$$

But $m-\alpha \geqq 0$ and $q-r<0$ because of the inequalities satisfied by the indices in (103.1). Hence $(m-\alpha)-(q-r)>0$ and

$$\nu \frac{a_\alpha}{a_r}\frac{b_q}{b_m} \geqq \nu\gamma \geqq \delta\gamma > Q.$$

Now put ρ and σ each equal to $P\gamma^{m+1}$. Then

$$a_\alpha = P\gamma^{n-\alpha+1} > P, \quad b_m = P\gamma^{n-m+1} > P,$$

which completes the proof of the lemma.

Consider the expression

$$\mu\{1-\Sigma P[\mathfrak{A}_{uv}-(\mathfrak{A}_{uv})_0]\}^{-1},$$

where the positive constants $P(>1)$ and $\mu(>1)$ are chosen so that this expression dominates the coefficients of the $C^\alpha_{\beta\gamma}$ in the ☆ terms of equations (103.1). Then if the constants $a_1, \ldots, a_n$ and $b_0, \ldots, b_n$ are chosen as in the above lemma, we have $a_\alpha/b_m \geqq 1$ and hence the coefficients of the second summation in equations (104.8) will dominate the coefficients of the corresponding summation in the ☆ terms of equations (103.1). Also if we take Q equal to the numerically greatest of the coefficients of the derivatives in the summation in equations (103.1), we see by the above lemma that these latter coefficients are less in absolute value than the corresponding coefficients in the first summation in equations (104.8). Then, since $D(1-\epsilon)a_\alpha/b_m \geqq 0$, it follows from (104.5) and (104.7) for $s=1$ that*

(104.10) $$\mathfrak{A}_{lm\,|\,\alpha_1}(0) \geqq |(A_{lm,\alpha_1})_0|,$$

where $\alpha_1 \leqq m$. If the indices $\alpha, \beta, \gamma, \delta$ in (104.1b) determine an independent component $A^\alpha_{\beta\gamma\delta}$ or A_{lm}, then we have from (104.6) that

(104.11) $$\mathfrak{C}^\alpha_{\beta\gamma\,|\,\delta}(0) \geqq |(A^\alpha_{\beta\gamma\delta})_0|,$$

since all components A_{lm} occur in the first summation in the right members of (104.1b). The inequality continues to hold if the indices $\alpha, \beta, \gamma, \delta$ determine a dependent component $A^\alpha_{\beta\gamma\delta}$, which is then equal to $-2A_{uv}$ or $-(A_{uv}+A_{pq})$ as can readily be observed. The inequality (104.11) is therefore satisfied for all values of the indices involved. In order to extend (104.11) to higher derivatives we assume the inequalities

(104.12) $$\mathfrak{C}^\alpha_{\beta\gamma\,|\,\delta_1\ldots\delta_s}(0) \geqq |(A^\alpha_{\beta\gamma\delta_1\ldots\delta_s})_0|$$

for $s<r(\geqq 2)$. If we compare successively equations (103.2) with the corresponding equations obtained from (104.8) by differentiation and elimination, it is then easily seen that

(104.13) $$\mathfrak{A}_{lm\,|\,\alpha_1\ldots\alpha_s}(0) \geqq |(A_{lm,\alpha_1\ldots\alpha_s})_0|,$$

where $s=1, \ldots, r$; $\alpha_1<m$; and α_i, for $i>1$, is arbitrary. Combining (104.7) and (104.13), we conclude that these latter inequalities are satisfied for all values of the indices $l, m, \alpha_1, \ldots, \alpha_s$ involved. Now differentiate (104.1b) repeatedly $r-1$ times, evaluate at the point $y^\alpha=0$, and compare with equations (47.6); this comparison shows that

(104.14) $$R\mathfrak{C}^\alpha_{\beta\gamma\,|\,\delta_1\ldots\delta_r}(0) \geqq 2\,|(A^\alpha_{\beta\gamma\delta_1\ldots\delta_r})_0-(A^\alpha_{\beta\delta_1\gamma\ldots\delta_r})_0|$$

for all values of the indices involved, or

(104.15) $$R\mathfrak{C}^\alpha_{\beta\delta_1\,|\,\gamma\ldots\delta_r}(0) \geqq 2\,|(A^\alpha_{\beta\delta_1\gamma\ldots\delta_r})_0-(A^\alpha_{\delta_1\delta_2\beta\ldots\delta_r})_0|$$

by an interchange of indices in (104.14). Now $\mathfrak{C}^\alpha_{\beta\gamma\,|\,\delta_1}$ is equal to $(a_{\delta_1}/a_\gamma)\,\mathfrak{C}^\alpha_{\beta\delta_1\,|\,\gamma}$ and hence

$$\mathfrak{C}^\alpha_{\beta\gamma\,|\,\delta_1}(y) \geqq R\mathfrak{C}^\alpha_{\beta\delta_1\,|\,\gamma}(y).$$

It therefore follows that the derivative $\mathfrak{C}^\alpha_{\beta\gamma\,|\,\delta_1\ldots\delta_r}$ is greater than, or at least equal to, the left member of (104.15). Hence

(104.16) $$\mathfrak{C}^\alpha_{\beta\gamma\,|\,\delta_1\ldots\delta_r}(0) \geqq 2\,|(A^\alpha_{\beta\delta_1\gamma\ldots\delta_r})_0-(A^\alpha_{\delta_1\delta_2\beta\ldots\delta_r})_0|.$$

* We here introduce the obvious notation $(A_{lm,\alpha_1})_0$ or more generally $(A_{lm,\alpha_1\ldots\alpha_s})_0$ for the derivative at $y^\alpha=0$ of the components A_{lm}.

Since $R<1$ the inequality (104.14) will remain true if the derivative $\mathfrak{C}^{\alpha}_{\beta\gamma\,|\,\delta_1\ldots\delta_r}(0)$ alone stands in its left member. Adding corresponding members of this latter inequality and the inequality (104.16), we obtain

$$\mathfrak{C}^{\alpha}_{\beta\gamma\,|\,\delta_1\ldots\delta_r}(0) \geqq \left| (A^{\alpha}_{\beta\gamma\delta_1\ldots\delta_r})_0 - (A^{\alpha}_{\delta_1\delta_2\beta\ldots\delta_r})_0 \right| .$$

By (104.14) and this last inequality it is therefore clear that we can write

$$\mathfrak{C}^{\alpha}_{\beta\gamma\,|\,\delta_1\ldots\delta_r}(0) \geqq \left| (A^{\alpha}_{\beta\gamma\delta_1\ldots\delta_r})_0 - (A^{\alpha}_{\mu\nu\eta_1\ldots\eta_r})_0 \right| , \tag{104.17}$$

where $\mu, \nu, \eta_1, \ldots, \eta_r$ is any permutation of $\beta, \gamma, \delta_1, \ldots, \delta_r$. Now form all permutations of the indices $\mu, \nu, \eta_1, \ldots, \eta_r$ and add together the $\mathfrak{N}$ corresponding inequalities (104.17). This gives

$$\mathfrak{N}\mathfrak{C}^{\alpha}_{\beta\gamma\,|\,\delta_1\ldots\delta_r}(0) \geqq \left| \mathfrak{N}\,(A^{\alpha}_{\beta\gamma\delta_1\ldots\delta_r})_0 - \Sigma\,(A^{\alpha}_{\mu\nu\eta_1\ldots\eta_r})_0 \right| .$$

Hence (104.12) is true for $s=r$ since the summation in the right member of this inequality vanishes by (41.2). With the recurrence process thus established it is clear that (104.12) and (104.13) are satisfied for $s=1, 2, \ldots$ and that (104.13) is in fact satisfied for all values of the indices $l, m, \alpha_1, \ldots, \alpha_s$ involved. Hence the power series expansions of the functions $\mathfrak{C}^{\alpha}_{\beta\gamma}(y)$ dominate the corresponding series (46.3) for the $C^{\alpha}_{\beta\gamma}$ with the result that the latter series converge. Similarly the power series expansions for the functions $\mathfrak{A}_{lm}$ dominate the series (103.3) for the independent components $A^{\alpha}_{\beta\gamma\delta}$ so that these converge. Since any dependent component $A^{\alpha}_{\beta\gamma\delta}$ is linearly dependent on the components A_{lm}, the convergence of the series for the dependent components $A^{\alpha}_{\beta\gamma\delta}$ follows immediately.

105. Convergence of the g series

By a procedure analogous to that employed in § 104 we can prove the convergence of the series (103.4) within a sufficiently small neighbourhood of the values $y^{\alpha}=0$. There is, however, enough difference in several of the details of this proof to warrant at least an indication of the method. We begin by considering the complete system of total differential equations

$$\frac{b_m}{a_\alpha}\frac{\partial \mathfrak{B}_{lm}}{\partial y^{\alpha}} = DF(y) + \frac{\mu\Theta(\mathfrak{B})}{1-\epsilon}\Sigma\mathfrak{C}^{\epsilon}_{\beta\gamma}, \tag{105.1a}$$

$$\frac{1}{a_\delta}\frac{\partial \mathfrak{C}^{\alpha}_{\beta\gamma}}{\partial y^{\delta}} = K\Xi(\mathfrak{H})\,\Sigma b_v\mathfrak{B}_{uv} + \frac{4}{R}\Sigma\mathfrak{C}^{\lambda}_{\mu\nu}\,\mathfrak{C}^{\rho}_{\sigma\tau}, \tag{105.1b}$$

$$\frac{1}{a_\gamma}\frac{\partial \mathfrak{H}_{\alpha\beta}}{\partial y^{\gamma}} = W\Sigma\mathfrak{H}_{\sigma\epsilon}\mathfrak{C}^{\epsilon}_{\nu\tau}. \tag{105.1c}$$

We observe immediately that (105.1a) can be written in the alternative form

$$\frac{\partial \mathfrak{B}_{lm}}{\partial y^{\alpha}} = \Sigma\nu\frac{a_\alpha}{a_r}\frac{b_q}{b_m}\frac{\partial \mathfrak{B}_{pq}}{\partial y^{r}} + \frac{a_\alpha}{b_m}D(1-\epsilon)F(y) + \frac{a_\alpha}{b_m}\mu\Theta(\mathfrak{B})\Sigma\mathfrak{C}^{\epsilon}_{\beta\gamma} \tag{105.2}$$

corresponding to (104.8), the positive constant $\epsilon\,(<1)$ being thus defined as the sum of m positive constants $\nu_1+\ldots+\nu_m$ as in the affine theory. The function $F(y)$ is defined by (104.3) where the positive constants Ω, ρ are chosen so that the expression (104.4) is dominant for each derivative $\partial\theta_{lm}/\partial y^{\alpha}$ $(\alpha>m)$ of the B_m functions θ_{lm} introduced in § 103. For the function $\Theta(\mathfrak{B})$ we can take

$$\{1-\Sigma b_v[\mathfrak{B}_{uv}-(\mathfrak{B}_{uv})_0]\}^{-1},$$

where the constants $a_\alpha\,(>1)$, $b_m\,(>1)$ and μ are given such values, in accordance with the lemma of § 104, that the function $\mu\Theta$ dominates the coefficients of the $C^{\alpha}_{\beta\gamma}$ in the equations corresponding to (103.1) for the metric case. The function

$$\Xi(\mathfrak{H}) = \frac{\Omega'}{1-\{\Sigma[\mathfrak{H}_{\alpha\beta}-(\mathfrak{H}_{\alpha\beta})_0]\}/\rho'},$$

in which Ω' and ρ' are positive constants, is dominant for each of the functions $g^{\alpha\beta}$, considered as rational functions of the $g_{\alpha\beta}$. The positive constants D, K, R, and W are to be taken such that

$$D \geqq \frac{b_m}{a_\alpha};\quad K \geqq \frac{16n}{aR};\quad R \leqq \frac{a_\beta}{a_\gamma},\quad R<1;\quad 1<a\leqq a_\gamma;\quad W \geqq \frac{2}{a},$$

where a is the least of the constants a_α. Finally we observe that the indices u, v of a component $\mathfrak{B}_{uv}$ assume all values that can be taken by the corresponding indices on the components B_{uv}; that other indices in the above equations assume values from 1 to n inclusive; and that all summations denote the sum of all terms that can be formed from the representative term by giving all possible values to the indices in question.

Consider a set of integrals of the system (105.1) whose initial values

$$(\mathfrak{B}_{lm})_0; \quad (\mathfrak{C}^\alpha_{\beta\gamma})_0; \quad (\mathfrak{H}_{\alpha\beta})_0 \quad \text{or} \quad \mathfrak{B}_{lm}(0); \quad \mathfrak{C}^\alpha_{\beta\gamma}(0); \quad \mathfrak{H}_{\alpha\beta}(0)$$

are such that

$$\begin{aligned} &\mathfrak{B}_{lm}(0) \geqq |(B_{lm})_0|, \\ &\mathfrak{C}^\alpha_{\beta\gamma}(0) \geqq 0, \\ &\mathfrak{H}_{\alpha\beta}(0) \geqq |(g_{\alpha\beta})_0|. \end{aligned}$$

The integrals of this set dominate the corresponding power series expansions of § 103. To prove this let us first observe that, as in the affine theory, we have

$$\mathfrak{B}_{lm|\alpha}(0) \geqq |(B_{lm,\,\alpha})_0|. \tag{105.3}$$

Taking account of the identity

$$A^\alpha_{\beta\gamma\delta} = g^{\alpha\sigma}(g_{\sigma\beta,\,\gamma\delta} + g_{\sigma\gamma,\,\delta\beta}), \tag{105.4}$$

easily derivable from (49.6) and (49.9), it follows immediately from (105.1b) that

$$\mathfrak{C}^\alpha_{\beta\gamma|\delta}(0) \geqq |(A^\alpha_{\beta\gamma\delta})_0|. \tag{105.5}$$

Differentiating (105.1c) and comparing with the corresponding equations (46.7), we obtain

$$\mathfrak{H}_{\alpha\beta|\gamma\delta}(0) \geqq |(g_{\alpha\beta,\,\gamma\delta})_0| \tag{105.6}$$

in virtue of (105.4). In order to extend the above inequalities (105.3), (105.5) and (105.6) to higher derivatives, we assume

$$\mathfrak{H}_{\alpha\beta|\gamma_1\ldots\gamma_s}(0) \geqq |(g_{\alpha\beta,\,\gamma_1\ldots\gamma_s})_0| \tag{105.7}$$

for $s \leqq r\,(\geqq 2)$; in addition we assume the inequalities (104.12) for $s<r\,(\geqq 2)$. By successive differentiation of (105.2) and comparison of the resulting equations with the equations corresponding to (103.2) for the metric case, we deduce

$$\mathfrak{B}_{lm|\alpha_1\ldots\alpha_s}(0) \geqq |(B_{lm,\,\alpha_1\ldots\alpha_s})_0| \tag{105.8}$$

for $s=1, \ldots, r$ and all values of the indices $l, m, \alpha_1, \ldots, \alpha_s$ involved. As in § 104 it now follows that for $s=r$ the inequalities (104.12) are likewise satisfied. By r-fold differentiation of (105.1c) we can therefore deduce that (105.7) is true for $s=r+1$. As this establishes the recurrence process we conclude that (104.12), (105.7) and (105.8) are satisfied for all values of the integer s. Hence it follows as in the affine case that the series (103.4) converge. Incidentally we have given a direct proof of the convergence of the power series expansion for the components $C^\alpha_{\beta\gamma}(y)$ of affine connection and for the components $\psi_{\alpha\beta}(y)$ of the fundamental metric tensor in normal coordinates.

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(2) Ref. (4), Chapter VI.

(3) The results of §§ 101–103 are a combination of those given by T. Y. Thomas, ref. (4), Chapter VI, and "Space structure as a boundary value problem", *Ann. of Math.* (2), **31** (1930), pp. 717–22.

(4) The method of proof used in §§ 104 and 105 is a combination of that used in § 47 and that employed by Méray and Riquier, "Sur la convergence des développements des intégrales ordinaires d'un système d'équations différentielles partielles", *Ann. l'éc. norm. sup.* (3), **7** (1890), pp. 46–55. Cf. also T. Y. Thomas, "Invariantive systems of partial differential equations", *Ann. of Math.* (2), **31** (1930), pp. 687–713.

INDEX

(The numbers refer to pages)

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